OPTIMUM THICKNESS OF THREE-LAYER SHRINK FITTED COMPOUND CYLINDER FOR UNIFORM STRESS DISTRIBUTION

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ABSTRACT

This paper introduces the optimum design for minimization of thickness of three-layer shrink-fitted compound cylinder to get equal maximum hoop stresses in all the cylinders. In the shrink-fitting problems, considering long hollow cylinders, the plane strain hypothesis can be regarded as more natural. Generally, hoop stress distribution across the wall of thick cylinder is non-linear in nature from inner to outer radius of the cylinder. The stresses reduce sharply towards the outer radius. Material is not utilized properly giving unnecessary more thickness. Hence two or more cylinders can be compounded by shrinkage process where outer cylinder is heated until it will slide freely over inner cylinder thus exerting the required contact pressure on cooling. As a result of shrinking, stress redistribution occurs across the wall of the compound cylinder and reduces the hoop stress and makes it more or less uniform over the thickness. In this paper effort is made to find optimum minimum thicknesses of three cylinders so that material volume is reduced and hoop stress is equal in all the cylinders. The analytical results of optimum design calculated with computer programming are validated in comparison with Finite Element Analysis in ANSYS Workbench. Both the results agree with each other. Hence this methodology can be applied for real-world mechanical applications of multi-layer compound cylinders.

KEYWORDS: *Residual Stress, Contact (Shrinkage) Pressure, Multi-layer Cylinder, Optimum Thickness*

I. INTRODUCTION

Multilayer compound cylinders are widely used in the field of high pressure technology such as hydraulic presses, forging presses, power plants, gas storages, chemical and nuclear plants, military applications etc. To enhance load bearing capacity and life of multilayer pressure vessels, different processes such as shrink fit and autofrettage are usually employed. Shrink fit increases load capacity.
Many researchers have focused on methods to extend lifetimes of vessels. Majzoobi et al. have proposed the optimization of bi-metal compound cylinders and minimized the weight of compound cylinder for a specific pressure [1]. The variables were shrinkage radius and shrinkage tolerance. Patil S. A. has introduced optimum design of two layer compound cylinder and optimized intermediate, outer diameter and shrinkage tolerance to get minimum volume of two layer compound cylinders [2-3]. Hamid Jahed et al. have investigated the optimum design of a three-layered vessel for maximum fatigue life expectancy under the combined effects of autofrettage and shrink fit [4]. Miraje Ayub A. and Patil Sunil A. have found minimum volume of three-layer open type compound cylinder considering plane stress hypothesis [5]. Yang Qiu-Ming et al. have presented a simple and visual tool to calculate the residual stress and describe the distribution of residual stress for both the elastic-perfectly plastic model and the strain-hardening mode [6].
To increase the pressure capacity of thick-walled cylinders, two or more cylinders (multi-layer) are shrunk into each other with different diametric differences to form compound cylinder. In three-layer compound cylinder the usual practice is to shrink the outer cylinder 3 on to the intermediate cylinder 2 and then shrink the resulting compound cylinder on to the inner cylinder 1 as shown in figure 1.

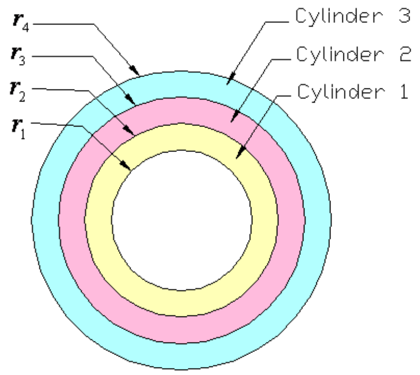


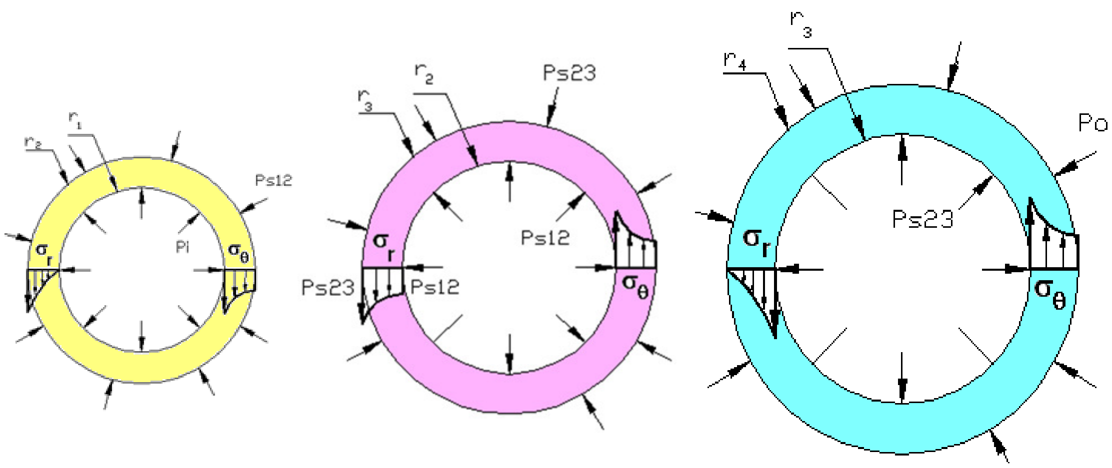
Figure 1. Three Layer Compound Cylinder

When the outer cylinder contracts on cooling the inner cylinder is brought into a state of compression. The outer cylinder will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage, thus a much smaller total fluctuation of hoop stress is obtained.

Designer usually face the problem of determining maximum hoop stress or maximum working stress to which shrunk-fit cylinder is subjected, the magnitude of shrinkage pressure or any combination. The design of shrunk-fit cylinders is much simplified if each cylinder is considered as a separate cylinder subjected simultaneously to the shrinkage pressure and the working pressure. Manufacturing and assembly process of such cylinders of real-world applications introduces some residual stresses. These residual stresses can be summed up with hoop stress developed due to internal pressure to find maximum hoop stress in all the cylinders.

The paper is organized as follows: We have explained Lame's theory as applied to compound cylinder in section 2.Optimum thickness methodology for compound cylinder and analytical results in section 3. Validation using FEM and ANSYS results in section 4. Discussion in section 5 followed by Conclusion in section 6.

II. LAME'S THEORY APPLIED TO COMPOUND CYLINDER



a) Cylinder 1 b) Cylinder 2 c) Cylinder 3

Figure 2. Radial and hoop stress distribution in three separate cylinders

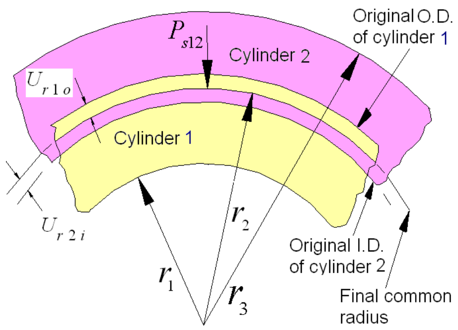


Figure 3. Interference between cylinder 1 & 2

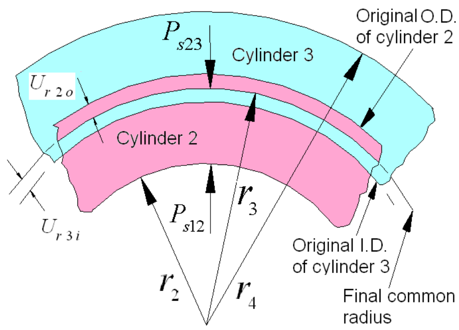


Figure 4. Interference between cylinder 2 & 3

Nomenclature

P_i	Internal pressure acting on the cylinder 1
P_o	External pressure on the cylinder 3
σ_θ	Hoop stress in the cylinder
σ_r	Radial stress in the cylinder
σ_z	Longitudinal (Axial) stress in the cylinder
r_1	Inner radius of cylinder 1
r_2	Outer radius of cylinder 1 and Inner radius of cylinder 2
r_3	Outer radius of cylinder 2 and Inner radius of cylinder 3
r_4	Outer radius of cylinder 3
P_{s12}	Contact pressure between cylinder 1 and 2
P_{s23}	Contact pressure between cylinder 2 and 3
$\varepsilon_{\theta 1o}$	Hoop strain in the outer wall of cylinder 1
U_{r1o}	Radial displacement at outer wall of cylinder 1
$\varepsilon_{\theta 2i}$	Hoop strain in the inner wall of cylinder 2
U_{r2i}	Radial displacement at inner wall of cylinder 2
δ_{12}	Total interference at the contact between cylinder 1 and 2
$\varepsilon_{\theta 2o}$	Hoop strain in the outer wall of cylinder 2
U_{r2o}	Radial displacement at outer wall of cylinder 2
$\varepsilon_{\theta 3i}$	Hoop strain in the inner wall of cylinder 3
U_{r3i}	Radial displacement at inner wall of cylinder 3
δ_{23}	Total interference at the contact between cylinder 2 and 3
ν	Poisson's ratio
t_1	r_2 / r_1
t_2	r_3 / r_2
t_3	r_4 / r_3

Consider three cylinders have same material. The method of solution for compound cylinders constructed from similar materials is to break the problem down into four separate effects:

i) shrinkage pressure P_{s12} only on the cylinder 1
ii) shrinkage pressure P_{s12} and P_{s23} only on the cylinder 2
iii) shrinkage pressure P_{s23} only on the cylinder 3
iv) internal pressure P_i only on the complete cylinder

Thus for each condition the hoop and radial stresses at any radius can be evaluated.

2.1 Radial and Hoop Stress in Cylinder 1

If $P_i = 0$ i. e. no internal pressure, radial stress in cylinder 1 is given by using Lame's equation

$$\sigma_r = -P_{s12}\frac{r_2^2}{r_2^2 - r_1^2}\left(1 - \frac{r_1^2}{r^2}\right) \tag{1}$$

σ_r is maximum at outer radius r_2 of cylinder 1 . Using equation (1)

$$\sigma_{r\max(at\ r_2)} = -P_{s12} \tag{2}$$

Hoop stress in cylinder 1 is given by using Lame's equation

$$\sigma_\theta = -P_{s12}\frac{r_2^2}{r_2^2 - r_1^2}\left(1 + \frac{r_1^2}{r^2}\right) \tag{3}$$

Hoop stress at outer radius r_2 is

$$\sigma_{\theta(at\ r_2)} = -P_{s12}\left[\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right] \tag{4}$$

While hoop stress at inner radius r_1 is

$$\sigma_{\theta\max(at\ r_1)} = -\left[\frac{2P_{s12}r_2^2}{r_2^2 - r_1^2}\right] \tag{5}$$

In the shrink-fitting problems, considering long hollow cylinders, the plane strain hypothesis (in general, $\sigma_z \neq 0$) can be regarded as more natural. Hence as per the relation

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right)$$

the expression for the hoop strain is given by

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\sigma_r - \nu\sigma_z\right] = \frac{1}{E}\left[\sigma_\theta - \nu\sigma_r - \nu^2(\sigma_r + \sigma_\theta)\right] = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right].$$

Using equations (2) and (4), assuming plane strain condition the hoop strain at the outer wall r_2 of cylinder 1 is

$$\varepsilon_{\theta 1o} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\left(-P_{s12}\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right) - \nu\left(-P_{s12}\right)\right] = \frac{U_{r1o}}{r_2} \tag{6}$$

Radial displacement U_{r1o} is

$$U_{r1o} = \frac{-P_{s12}r_2\left(1+\nu\right)}{E}\left[(1-\nu)\left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right) - \nu\right] \tag{7}$$

2.2 Radial and Hoop Stress in Cylinder 2

Contact pressure P_{s12} is acting as internal pressure and contact pressure P_{s23} is acting as external pressure on cylinder 2.

Using Lame's equation, radial stress in the cylinder 2 at inner radius r_2 is given by

$$\sigma_{r(at\ r_2)} = -P_{s12} \tag{8}$$

While radial stress in the cylinder 2 at outer radius r_3 is given by

$$\sigma_{r(at\ r_3)} = -P_{s23} \tag{9}$$

Hoop stress in the cylinder 2 at inner radius r_2 is given by

$$\sigma_{\theta\max(at\ r_2)} = \frac{P_{s12}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} - \frac{2P_{s23}(r_3^2\ )}{r_3^2 - r_2^2} \tag{10}$$

While hoop stress in the cylinder 2 at outer radius r_3 is given by

$$\sigma_{\theta(at\ r_3)} = \frac{2P_{s12}\ (r_2^2)}{r_3^2 - r_2^2} - \frac{P_{s23}(r_3^2 + r_2^2)}{r_3^2 - r_2^2} \tag{11}$$

Using equations (8) and (10), assuming plane strain condition the hoop strain at the inner wall r_2 of cylinder 2 is

$$\varepsilon_{\theta 2i} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\left(P_{s12}\frac{r_3^2+r_2^2}{r_3^2-r_2^2} - \frac{2P_{s23}r_3^2}{r_3^2-r_2^2}\right) - \nu\left(-P_{s12}\right)\right] = \frac{U_{r2i}}{r_2} \quad (12)$$

Radial displacement U_{r2i}

$$U_{r2i} = \frac{r_2(1+\nu)}{E}\left[P_{s12}\left((1-\nu)\frac{r_3^2+r_2^2}{r_3^2-r_2^2}+\nu\right) - (1-\nu)\left(\frac{2P_{s23}r_3^2}{r_3^2-r_2^2}\right)\right] \quad (13)$$

Referring figure 3 and using equations (7) and (13), total interference δ_{12} at the contact between cylinder 1 and 2 is

$$\delta_{12} = U_{r2i} - U_{r1o}$$

$$= \frac{r_2(1+\nu)}{E}\left[P_{s12}\left((1-\nu)\frac{r_3^2+r_2^2}{r_3^2-r_2^2}+\nu\right) - (1-\nu)\left(\frac{2P_{s23}r_3^2}{r_3^2-r_2^2}\right)\right] - \frac{-P_{s12}r_2(1+\nu)}{E}\left[(1-\nu)\left(\frac{r_2^2+r_1^2}{r_2^2-r_1^2}\right)-\nu\right]$$

$$= \frac{r_2(1-\nu^2)}{E}\left[P_{s12}\left(\frac{r_3^2+r_2^2}{r_3^2-r_2^2}+\frac{r_2^2+r_1^2}{r_2^2-r_1^2}\right) - 2P_{s23}\left(\frac{r_3^2}{r_3^2-r_2^2}\right)\right] \quad (14)$$

Using equations (9) and (11), hoop strain in the outer wall r_3 of cylinder 2 is given by

$$\varepsilon_{\theta 2o} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\left(\frac{2P_{s12}(r_2^2)}{r_3^2-r_2^2} - \frac{P_{s23}(r_3^2+r_2^2)}{r_3^2-r_2^2}\right) - \nu\left(-P_{s23}\right)\right] = \frac{U_{r2o}}{r_3} \quad (15)$$

Hence radial displacement U_{r2o}

$$U_{r2o} = \frac{r_3}{E}\left[\frac{2P_{s12}(1-\nu^2)r_2^2}{r_3^2-r_2^2} - P_{s23}(1+\nu)\left((1-\nu)\frac{r_3^2+r_2^2}{r_3^2-r_2^2}-\nu\right)\right] \quad (16)$$

2.3 Radial and Hoop Stress in Cylinder 3

Contact pressure P_{s23} is acting as internal pressure on cylinder 3 and external pressure P_o is zero.

Radial stress in the cylinder 3 at inner radius r_3 is given by

$$\sigma_{r(at\ r_3)} = -P_{s23} \quad (17)$$

Hoop stress in the cylinder 3 at inner radius r_3 is given by

$$\sigma_{\theta\ max(at\ r_3)} = \frac{P_{s23}(r_4^2+r_3^2)}{r_4^2-r_3^2} \quad (18)$$

While hoop stress in the cylinder 3 at outer radius r_4 is given by

$$\sigma_{\theta(at\ r_4)} = \frac{2P_{s23}(r_3^2)}{r_4^2-r_3^2} \quad (19)$$

Using equations (17) and (18), hoop strain at inner wall r_3 of cylinder 3 is given by

$$\varepsilon_{\theta 3i} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\left(\frac{P_{s23}(r_4^2+r_3^2)}{r_4^2-r_3^2}\right) - \nu\left(-P_{s23}\right)\right] = \frac{U_{r3i}}{r_3} \quad (20)$$

Radial displacement U_{r3i}

$$U_{r3i} = \frac{P_{s23}r_3(1+\nu)}{E}\left[(1-\nu)\left(\frac{r_4^2+r_3^2}{r_4^2-r_3^2}\right)+\nu\right] \quad (21)$$

Referring figure 4 and using equations (16) and (21), total interference δ_{23} at the contact between cylinder 2 and 3

$$\delta_{23} = U_{r3i} - U_{r2o}$$

$$= \frac{P_{s23}r_3(1+\nu)}{E}\left[(1-\nu)\left(\frac{r_4^2+r_3^2}{r_4^2-r_3^2}\right)+\nu\right] - \frac{r_3}{E}\left[\frac{2P_{s12}(1-\nu^2)r_2^2}{r_3^2-r_2^2} - P_{s23}(1+\nu)\left((1-\nu)\frac{r_3^2+r_2^2}{r_3^2-r_2^2}-\nu\right)\right]$$

$$= \frac{r_3\left(1-\nu^2\right)}{E}\left[P_{s23}\left(\frac{r_4^{\,2}+r_3^{\,2}}{r_4^{\,2}-r_3^{\,2}}+\frac{r_3^{\,2}+r_2^{\,2}}{r_3^{\,2}-r_2^{\,2}}\right)-\frac{2P_{s12}r_2^{\,2}}{r_3^{\,2}-r_2^{\,2}}\right] \qquad (22)$$

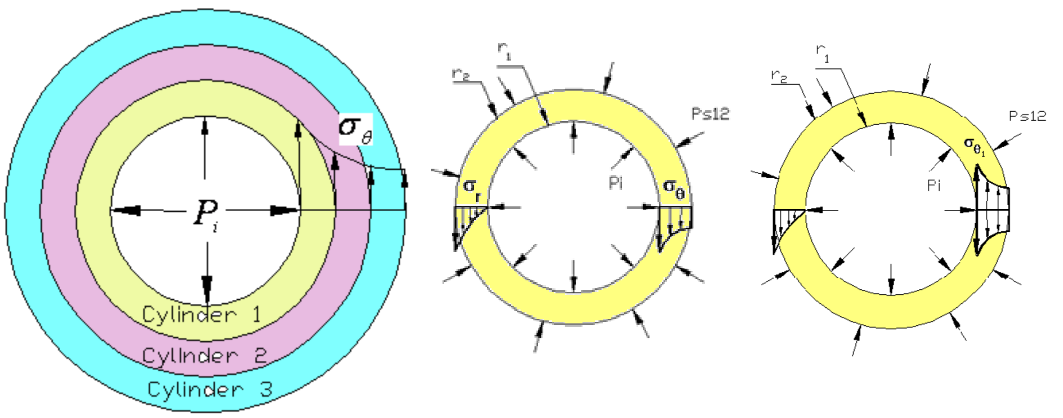
Hoop stress at any radius r in compound cylinder due to internal pressure only is given by

$$\sigma_\theta = \frac{P_i r_1^{\,2}}{r_4^{\,2}-r_1^{\,2}}\left[\frac{r_4^{\,2}}{r^2}+1\right] \qquad (23)$$

2.4 Principle of Superposition

After finding hoop stresses at all the radii, the principle of superposition is applied, i.e. the various stresses are then combined algebraically to produce the resultant hoop stresses in the compound cylinder subjected to both shrinkage pressures and internal pressure P_i .

2.4.1 Resultant hoop stress in Cylinder 1



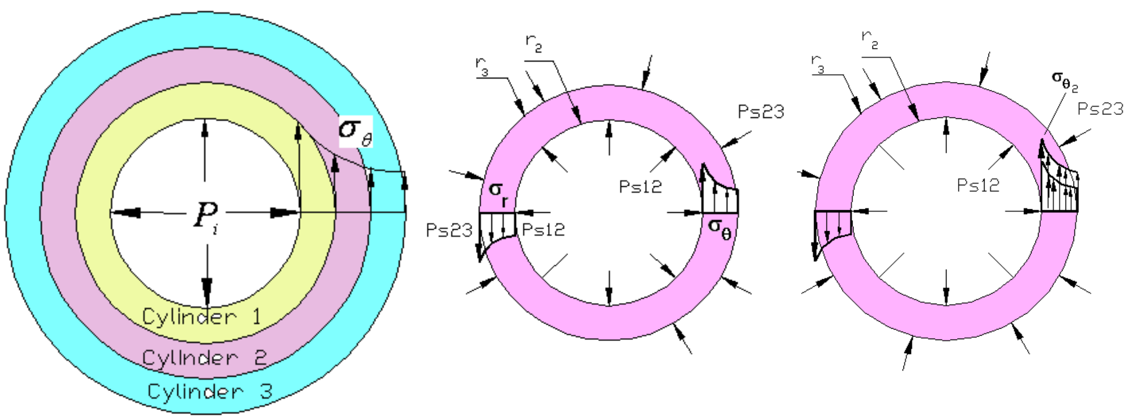
a) Hoop stress due to P_i b) Residual stress due to P_{s12} (c) Resultant stress $\sigma_{\theta 1}$

Figure 5. Superposition of hoop stress due to P_i & residual stress due to P_{s12} in cylinder 1

Using equations (23) and (5), maximum hoop stress at the inner surfaces of cylinder 1 at r_1

$$\sigma_{\theta 1} = P_i\left[\frac{r_4^{\,2}+r_1^{\,2}}{r_4^{\,2}-r_1^{\,2}}\right]-2P_{s12}\left[\frac{r_2^{\,2}}{r_2^{\,2}-r_1^{\,2}}\right] \qquad (24)$$

2.4.2 Resultant hoop stress in Cylinder 2



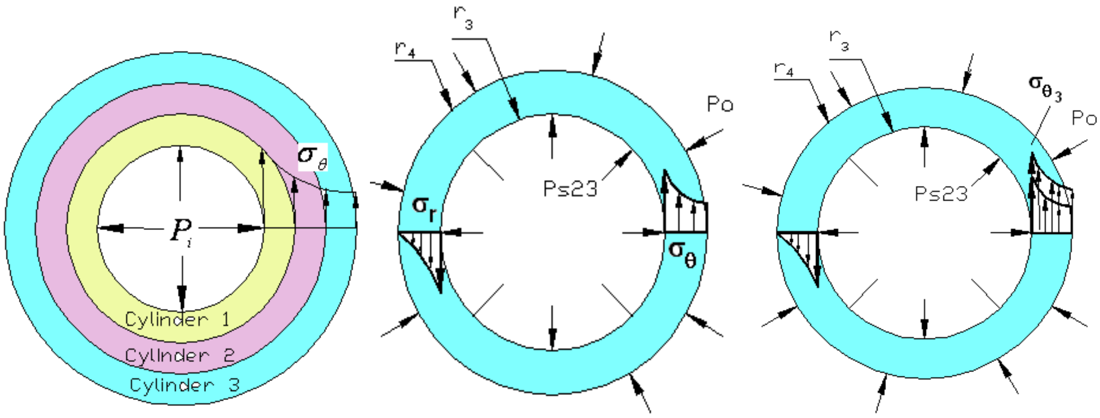
a) Hoop stress due to P_i b) Residual stress due to P_{s12} & P_{s23} (c) Resultant stress $\sigma_{\theta 2}$

Figure 6. Superposition of hoop stress due to P_i & residual stress due to P_{s12} & P_{s23} in cylinder 2

Using equations (23) and (10), maximum hoop stress at the inner surfaces of cylinder 2 at r_2

$$\sigma_{\theta 2} = \frac{P_i r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right] + \frac{P_{s12}(r_3^2 + r_2^2) - 2P_{s23} r_3^2}{r_3^2 - r_2^2} \quad (25)$$

2.4.3 Resultant hoop stress in Cylinder 3



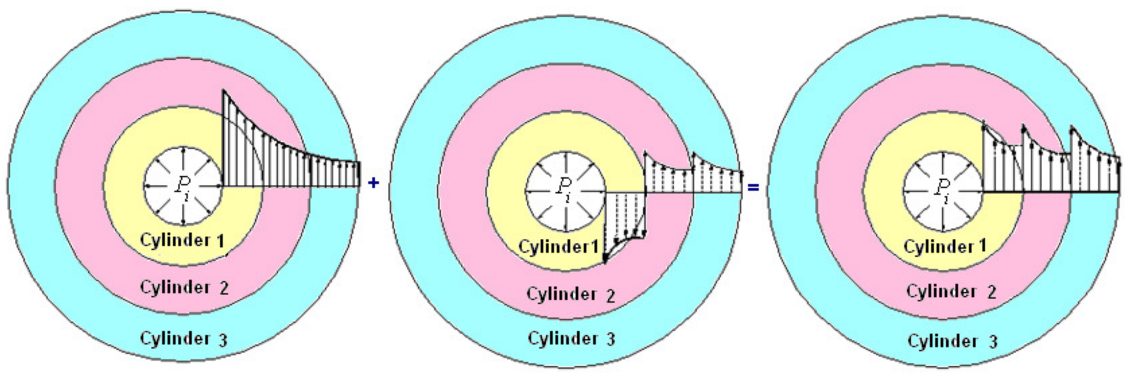
a) Hoop stress due to P_i b) Residual stress due to P_{s23} (c) Resultant stress $\sigma_{\theta 3}$

Figure 7. Superposition of hoop stress due to P_i & residual stress due to P_{s23} in cylinder 3

Using equations (18) and (23), maximum hoop stress at the inner surfaces of cylinder 3 at r_3

$$\sigma_{\theta 3} = \frac{P_i r_1^2}{r_3^2}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_1^2}\right] + P_{s23}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_3^2}\right] \quad (26)$$

2.4.4 Resultant hoop stress in Compound Cylinder



a) Hoop stress due to P_i b) Residual stress due to P_{s12} & P_{s23} (c) Resultant Hoop stress

Figure 8. Superposition of hoop stress due to P_i & residual stresses due to P_{s12} & P_{s23} in all cylinders

III. METHODOLOGY OF OPTIMUM THICKNESS FOR COMPOUND CYLINDER

To obtain optimum values of the contact (shrinkage) pressures P_{s12} and P_{s23} which will produce equal hoop (tensile) stresses in all the three cylinders, maximum hoop stresses given by the equations (24), (25) and (26) have been equated.

Equating equations (24) and (25) i. e. $\sigma_{\theta 1} = \sigma_{\theta 2}$ and rearranging,

$$P_{s12}\left[\frac{2r_2^2}{r_2^2 - r_1^2} + \frac{r_3^2 + r_2^2}{r_3^2 - r_2^2}\right] = P_i\left[\frac{r_4^2 + r_1^2}{r_4^2 - r_1^2} - \frac{r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right]\right] + P_{s23}\frac{2r_3^2}{r_3^2 - r_2^2} \quad (27)$$

Let the ratios $t_1 = \frac{r_2}{r_1} = \frac{d_2}{d_1}$, $t_2 = \frac{r_3}{r_2} = \frac{d_3}{d_2}$, $t_3 = \frac{r_4}{r_3} = \frac{d_4}{d_3}$ (28)

Where d_1, d_2, d_3, d_4 are diameters corresponding to radii r_1, r_2, r_3, r_4.

Hence $t_1 t_2 = \frac{r_2}{r_1}.\frac{r_3}{r_2} = \frac{r_3}{r_1}$, $t_2 t_3 = \frac{r_3}{r_2}\frac{r_4}{r_3} = \frac{r_4}{r_2}$, $t_1 t_2 t_3 = \frac{r_2}{r_1}.\frac{r_3}{r_2}\frac{r_4}{r_3} = \frac{r_4}{r_1}$

Let $$k_1 = \frac{2r_2^2}{r_2^2 - r_1^2} + \frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = \frac{2t_1^2}{t_1^2 - 1} + \frac{t_2^2 + 1}{t_2^2 - 1} \quad (29)$$

$$k_2 = \frac{r_4^2 + r_1^2}{r_4^2 - r_1^2} - \frac{r_1^2}{r_2^2}\left[\frac{r_4^2 + r_2^2}{r_4^2 - r_1^2}\right] = \frac{t_1^2 t_2^2 t_3^2 + 1}{t_1^2 t_2^2 t_3^2 - 1} - \frac{t_2^2 t_3^2 + 1}{t_1^2 t_2^2 t_3^2 - 1} \quad (30)$$

$$k_3 = \frac{2r_3^2}{r_3^2 - r_2^2} = \frac{2t_2^2}{t_2^2 - 1} \quad (31)$$

Hence equation (27) becomes

$$P_{s12} = P_i\left[k_2 / k_1\right] + P_{s23}\left[k_3 / k_1\right] \quad (32)$$

Equating equations (25) and (26) i. e. $\sigma_{\theta 2} = \sigma_{\theta 3}$ and rearranging,

$$P_{s12}\frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = P_i\left[\frac{r_1^2(r_4^2 + r_3^2)}{r_3^2(r_4^2 - r_1^2)} - \frac{r_1^2(r_4^2 + r_2^2)}{r_2^2(r_4^2 - r_1^2)}\right] + P_{s23}\left[\frac{r_4^2 + r_3^2}{r_4^2 - r_3^2} + \frac{2r_3^2}{r_3^2 - r_2^2}\right] \quad (33)$$

Let $$k_4 = \frac{r_3^2 + r_2^2}{r_3^2 - r_2^2} = \frac{t_2^2 + 1}{t_2^2 - 1} \quad (34)$$

$$k_5 = \frac{r_1^2(r_4^2 + r_3^2)}{r_3^2(r_4^2 - r_1^2)} - \frac{r_1^2(r_4^2 + r_2^2)}{r_2^2(r_4^2 - r_1^2)} = \frac{t_3^2 + 1}{t_1^2 t_2^2 t_3^2 - 1} - \frac{t_2^2 t_3^2 + 1}{t_1^2 t_2^2 t_3^2 - 1} \quad (35)$$

$$k_6 = \frac{r_4^2 + r_3^2}{r_4^2 - r_3^2} + \frac{2r_3^2}{r_3^2 - r_2^2} = \frac{t_3^2 + 1}{t_3^2 - 1} + \frac{2t_2^2}{t_2^2 - 1} \quad (36)$$

Hence equation (33) becomes

$$P_{s12} = P_i\left[k_5 / k_4\right] + P_{s23}\left[k_6 / k_4\right] \quad (37)$$

Equations (32) and (36) have been solved to get P_{s12} and P_{s23} in terms of P_i as follows,

$$P_{s12} = P_i\left[\frac{(k_5 / k_6) - (k_2 / k_3)}{(k_4 / k_6) - (k_1 / k_3)}\right] \quad (38)$$

$$P_{s23} = P_i\left[\frac{(k_5 / k_4) - (k_2 / k_1)}{(k_3 / k_1) - (k_6 / k_4)}\right] \quad (39)$$

Putting the values of t_1, t_2 and t_3, the equations (14) and (22) can be written as

$$\delta_{12} = \frac{r_2(1-\nu^2)}{E}\left[P_{s12}\left(\frac{t_2^2 + 1}{t_2^2 - 1} + \frac{t_1^2 + 1}{t_1^2 - 1}\right) - 2P_{s23}\left(\frac{t_2^2}{t_2^2 - 1}\right)\right] \quad (40)$$

$$\delta_{23} = \frac{r_3(1-\nu^2)}{E}\left[P_{s23}\left(\frac{t_3^2 + 1}{t_3^2 - 1} + \frac{t_2^2 + 1}{t_2^2 - 1}\right) - \frac{2P_{s12}}{t_2^2 - 1}\right] \quad (41)$$

3.1 Analytical Method

For the given volume of fluid to be stored, the internal diameter of cylinder 1 (d_1) is known. Here it is assumed as 100 mm. Material for all the three cylinders is assumed as same i. e. steel. Yield strength of the steel material is σ_y = 250 MPa. Maximum hoop stress is major stress which is responsible for failure of cylinder. Maximum principal stresses in all the cylinders (here it is maximum hoop stress) should not exceed the yield stress of the material to avoid the failure of the compound cylinder. Optimum material volume (assuming unit length) can be calculated using following steps.

1. Assume internal diameter of cylinder 1 (d_1) say 100 mm.
2. Select the ratios $t_1 = \frac{r_2}{r_1} = \frac{d_2}{d_1}$, $t_2 = \frac{r_3}{r_2} = \frac{d_3}{d_2}$, $t_3 = \frac{r_4}{r_3} = \frac{d_4}{d_3}$
3. For the given internal pressure P_i , one can find contact (shrinkage) pressures P_{s12} and P_{s23} in terms of ratios t_1 , t_2 and t_3 .
4. Find the volume of the compound cylinder using $Fx = \pi.(d_4^2 - d_1^2)/4$
5. Minimize the volume subjected to the constraints,
   i) $\sigma_{\theta 1} <= \sigma_y$ ii) $\sigma_{\theta 2} <= \sigma_y$ iii) $\sigma_{\theta 3} <= \sigma_y$ iv) $\delta_{12} > 0$ v) $\delta_{23} > 0$
6. Optimized parameters t_1 , t_2 , t_3 are used for the design.

In computer programming, the values of t_1 , t_2 and t_3 are selected from 1.1 to 2.4 with the increment of 0.10, 0.05 and refined up to 0.002. Thus with lot of combinations of t_1, t_2 & t_3 material volume is found. A number of combinations of t_1 , t_2 and t_3 satisfy the condition of equal maximum hoop stresses in all three cylinders which is less than yield stress of the material. Out of these combinations some important combinations are presented in this paper for comparison. However there is one unique combination where volume is minimum. In programming, equations (38) and (39) are used to find contact (shrinkage) pressures P_{s12} and P_{s23} for given internal pressure P_i resp. Also equations (40) and (41) are used to find interferences δ_{12}, δ_{23} resp. These interferences are later used in section 4 for modeling in Finite Element Method.

Algorithm of Computer Program

```
Assign values to d1, σy, E and Pi .
For (t1 1.10 to 2.4 with increment of 0.02)
  For (t2 1.10 to 2.4 with increment of 0.02)
    For (t3 1.10 to 2.4 with increment of 0.02)
    {
      Calculate k1, k2, k3, k4, k5 and k6 in terms of t1, t2, t3.
      Calculate Ps12 and Ps23 in terms of k1, k2, k3, k4, k5 and k6 .
      Calculate δ12 and δ23 in terms of Ps12, Ps23 and k1, k2, k3, k4, k5, k6 .
      Calculate maximum hoop stresses σθ1, σθ2, σθ3 in terms of Pi and t1, t2, t3
           Check the conditions σθ1 <= σy , σθ2 <= σy and σθ3 <= σy
                Calculate the volume Fx = π.(d4^2 − d1^2)/4
                     Compare new calculated volume with previous one.
    } // End of loops for t1 , t2 and t3
Print the parameterized values and "Minimum Volume".
```

3.2 Analytical Results

Results from computer programming are listed in table 1 and 2.

Table 1. Diameters and ratios of diameters and Volume for compound cylinder (assuming d_1 = 100 mm)

Combination	t_1	t_2	t_3	d_2	d_3	d_4	Fx
1	1.370	1.290	1.260	137.0	176.7	222.7	31090.30
2	**1.285**	**1.305**	**1.325**	128.5	167.7	222.2	**30920.07**
3	1.270	1.318	1.328	127.0	167.4	222.3	30953.17

Table 2. Maximum hoop stress, contact pressure and interference for compound cylinder

Combination	$\sigma_{\theta 1}$	$\sigma_{\theta 2}$	$\sigma_{\theta 3}$	P_{s12}	P_{s23}	δ_{12}	δ_{23}
1	249.94	249.94	249.94	29.5	19.6	0.038	0.032
2	250.00	250.00	250.00	25.0	20.6	0.027	0.030

3	250.00	250.00	250.00	24.1	20.6	0.026	0.031

The minimum volume of three-layer compound cylinder is **30920.07** mm^3 corresponding to values in combination 2 where ratios t_1 = **1.285**, t_2 = **1.305**, t_3 = **1.325**. Hence optimum thickness of cylinder 1 is **28.5** mm, cylinder 2 is **39.2** mm and cylinder is **54.5** mm.

IV. Validation using FEM and ANSYS Results

Using optimized parameters t_1 , t_2 , t_3 from the combination set number 2 of the table 1 and taking d_1 = 100 mm remaining diameters d_2, d_3, d_4 are calculated as follows.

$d_2 = t_1 \times d_1 = 128.5\text{mm}$, $d_3 = t_2 \times d_2 = 167.7\text{mm}$, $d_4 = t_3 \times d_3 = 222.2\text{mm}$.

The optimized values of δ_{12}, δ_{23} are radius based. These values are doubled to take diametric effect in the three dimensional Finite Element Model. Using these values, inner diameter of cylinder 2 (d_{2i}) and inner diameter of cylinder 3 (d_{3i}) for shrink fit are calculated as shown in table 3.

Table 3. Data for modeling in ANSYS (for combination set number 2)

t_1	t_2	t_3	d_2	d_{2i}	d_3	d_{3i}	d_4	δ_{12}	δ_{23}
1.285	1.305	1.325	128.500	128.446	167.700	167.640	222.20	0.027	0.030

where
d_1, d_2 = inner & outer diameters of cylinder 1 respectively.
d_{2i} , d_3 = inner & outer diameters of cylinder 2 respectively for shrink fit.
d_{3i}, d_4 = inner & outer diameters of cylinder 3 respectively for shrink fit.

Using the values of δ_{12}, δ_{23} shrink fit is applied between cylinders 1 & 2 and between cylinders 2 & 3 respectively in ANSYS Workbench. Contact between cylinders 1 & 2 as well as between cylinders 2 & 3 is applied using contact tool in ANSYS Workbench. Similarly, data for modeling is prepared for combination set numbers 1 as well as 3 and modeling is done in ANSYS Workbench.

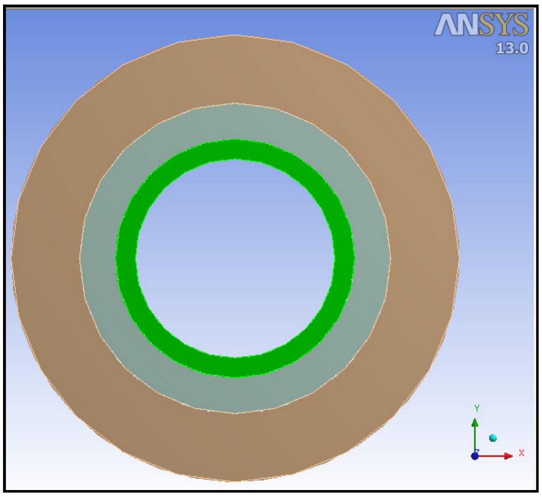


Figure 9. Three Layer Compound Cylinder

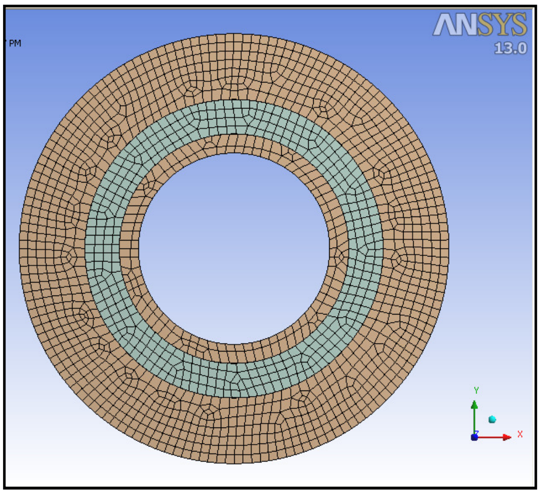


Figure 10. Finite Element Mesh

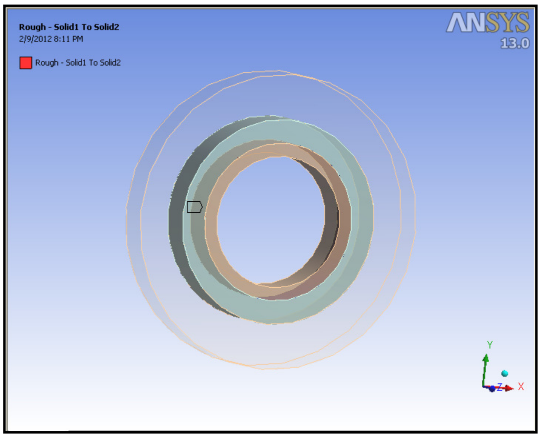


Figure 11. Shrink fit between cylinder 1 & 2

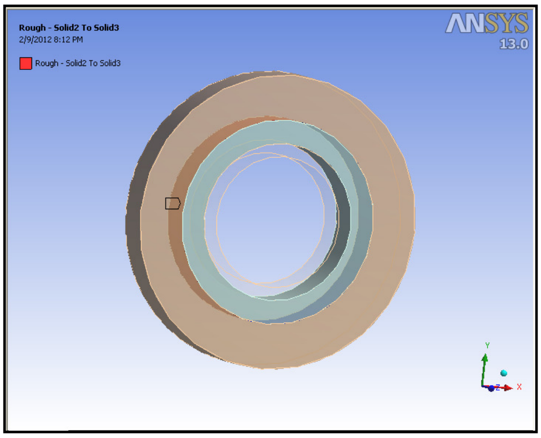


Figure 12. Shrink fit between cylinder 2 & 3

Results by ANSYS Workbench for set number 1 are shown in the figures 13 to 17.

<u>COMBINATION SET 1</u> (t_1 = **1.370**, t_2 = **1.290**, t_3 = **1.260**)

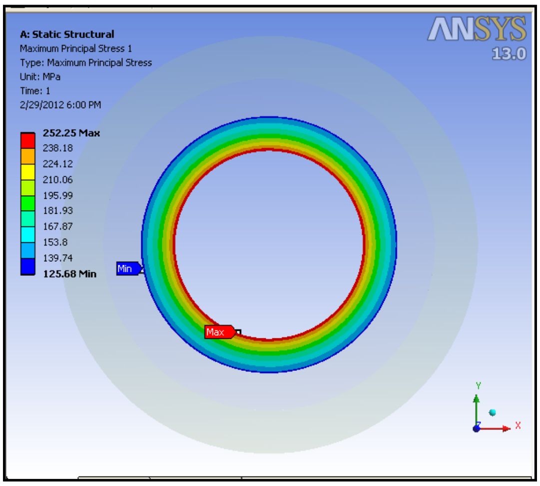


Figure 13. Maximum Principal Stress in cylinder 1

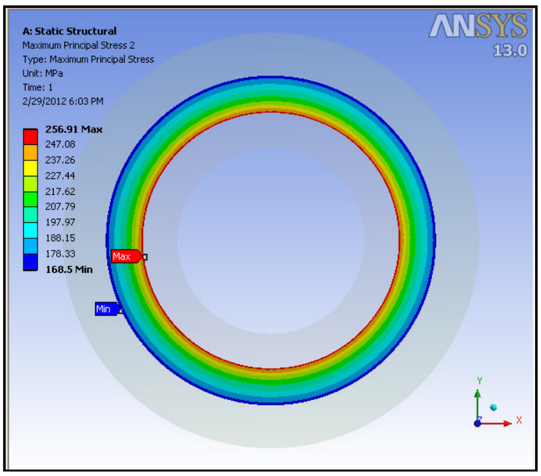


Figure 14. Maximum Principal Stress in cylinder 2

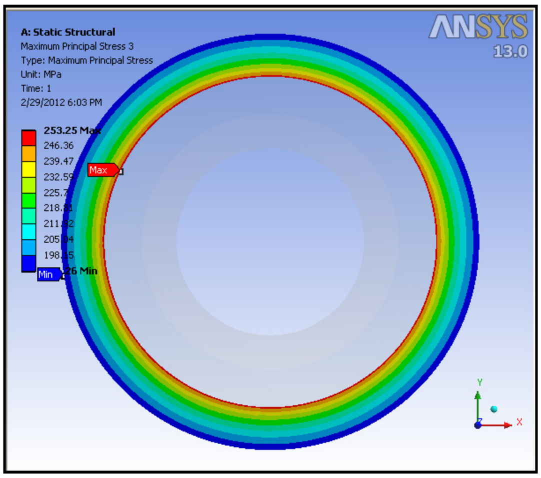


Figure 15. Maximum Principal Stress in cylinder 3

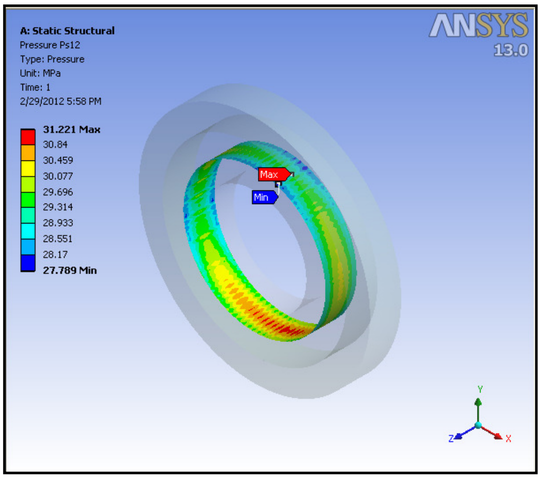


Figure 16. Contact pressure between cylinder 1 & 2
P_{s12} = 29.50 MPa (Avg) (without P_i)

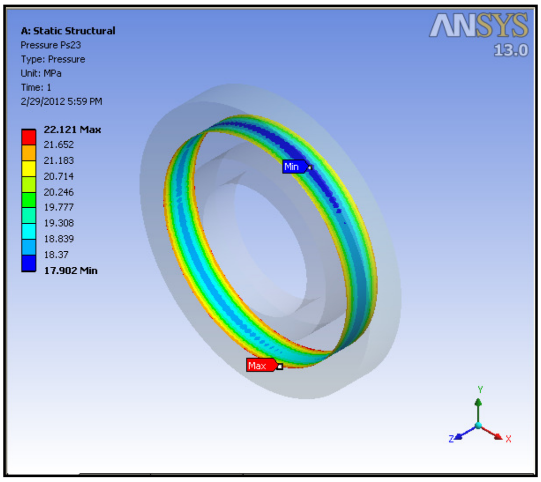


Figure 17. Contact pressure between cylinder 2 & 3
P_{s23} = 20.01 MPa (Avg) (without P_i)

Results by ANSYS Workbench for set number 2 are shown in the figures 18 to 22.

<u>COMBINATION SET 2</u> (t_1 = **1.285**, t_2 = **1.305**, t_3 = **1.325**)

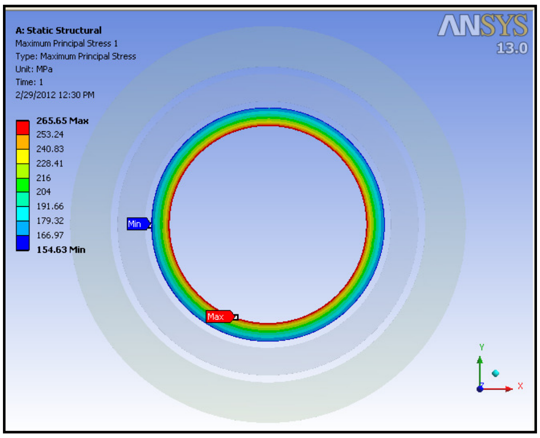


Figure 18. Maximum Principal Stress in cylinder 1

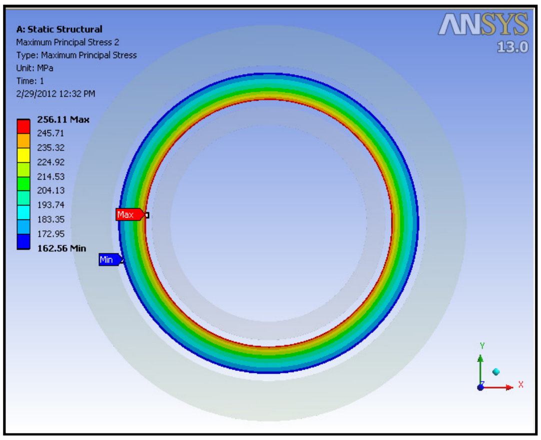


Figure 19. Maximum Principal Stress in cylinder 2

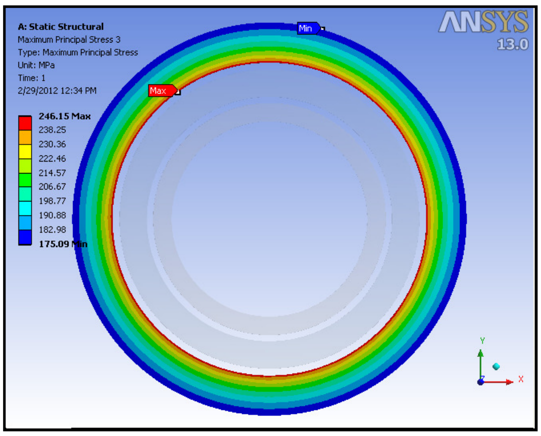


Figure 20. Maximum Principal Stress in cylinder 3

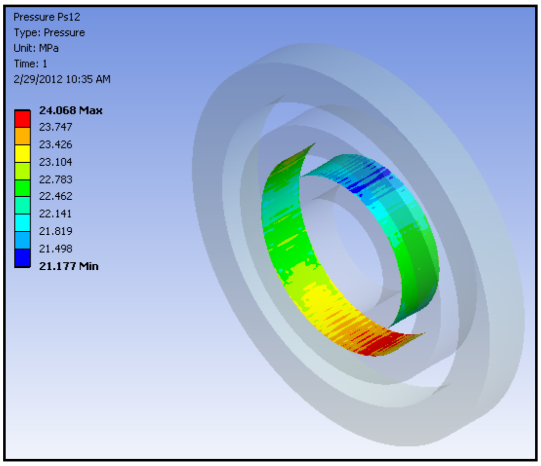


Figure 21. Contact pressure between cylinder 1 & 2
P_{s12} = 22.58 MPa (Avg) (without P_i)

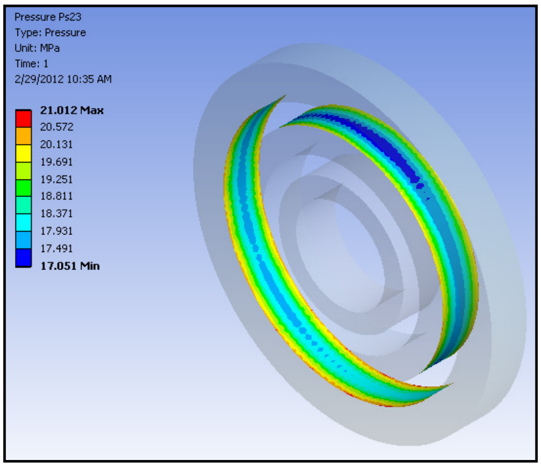


Figure 22. Contact pressure between cylinder 2 & 3
P_{s23} = 19.00 MPa (Avg) (without P_i)

Results by ANSYS Workbench for set number 3 are shown in the figures 23 to 27.

<u>COMBINATION SET 3</u> (t_1 = **1.270**, t_2 = **1.318**, t_3 = **1.328**)

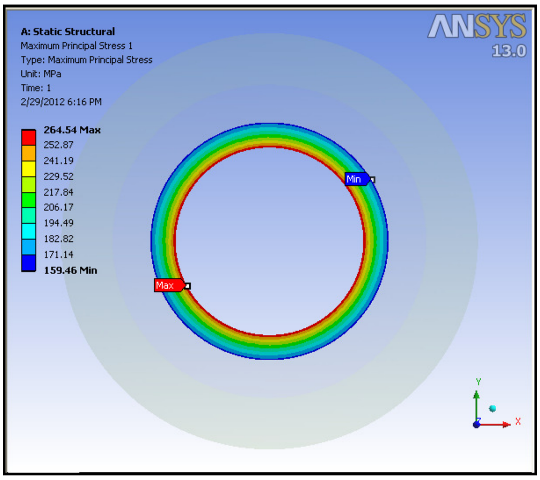


Figure 23. Maximum Principal Stress in cylinder 1

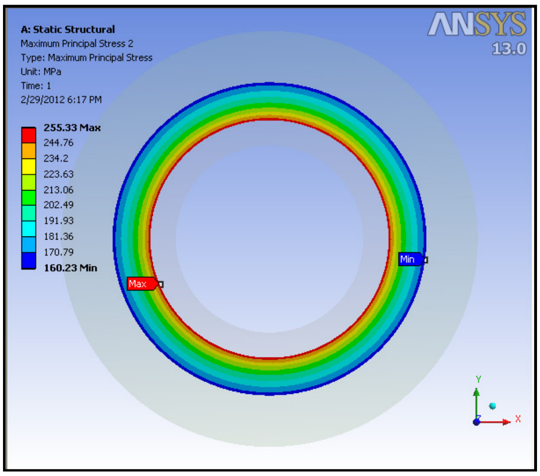


Figure 24. Maximum Principal Stress in cylinder 2

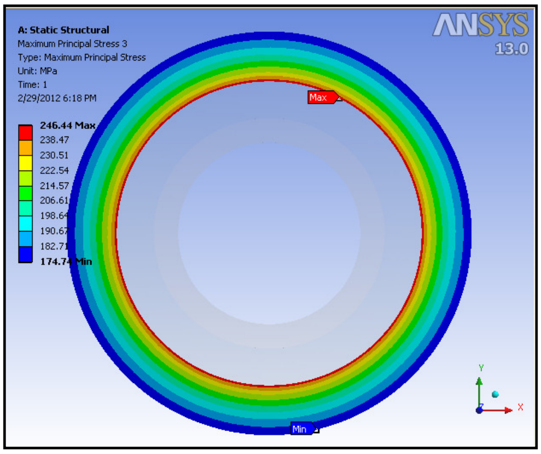


Figure 25. Maximum Principal Stress in cylinder 3

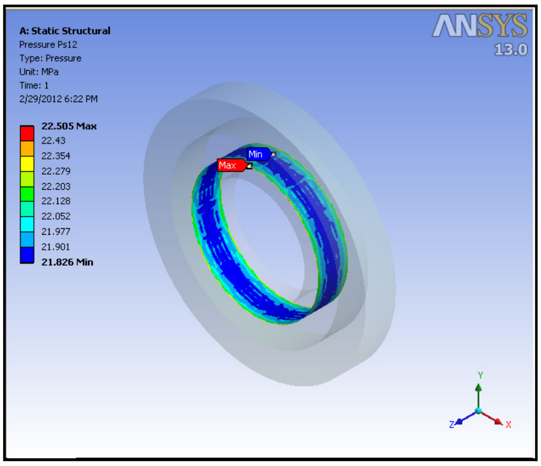


Figure 26. Contact pressure between cylinder 1 & 2 P_{s12} = 22.16 MPa (Avg) (without P_i)

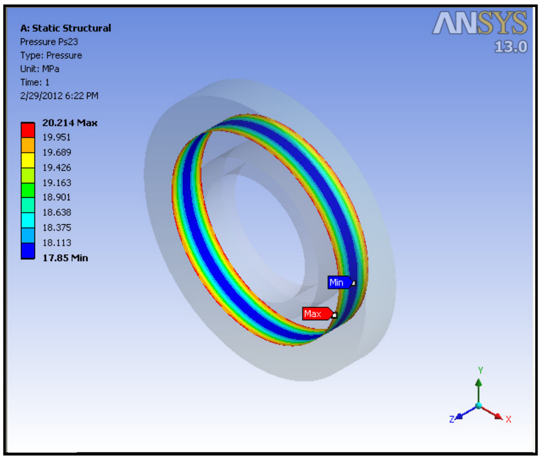


Figure 27. Contact pressure between cylinder 2 & 3 P_{s23} = 19.03 MPa (Avg) (without P_i)

V. DISCUSSION

Analytical results and FEM (ANSYS) results are summarized in Table 4 and 5.

Table 4. Comparison of hoop stresses by Analytical and ANSYS results

Combinations	Results	Maximum hoop stress in cylinder 1 $\sigma_{\theta 1}$ (MPa)	Maximum hoop stress in cylinder 2 $\sigma_{\theta 2}$ (MPa)	Maximum hoop stress in cylinder 3 $\sigma_{\theta 3}$ (MPa)
1	Analytical	249.94	249.94	249.94
	ANSYS	252.25	256.91	253.25
2	Analytical	250.00	250.00	250.00
	ANSYS	265.65	256.11	246.15
3	Analytical	250.00	250.00	250.00
	ANSYS	264.54	255.33	246.44

Table 5. Comparison of contact pressures by Analytical and ANSYS results

Combinations	Results	Contact pressure between cylinder 1 & 2 P_{s12} (MPa)	Contact pressure between cylinder 2 & 3 P_{s23} (MPa)
1	Analytical	29.50	19.60
	ANSYS	29.50	20.01
2	Analytical	25.00	20.60
	ANSYS	22.58	19.00
3	Analytical	24.10	20.60
	ANSYS	22.16	19.03

VI. CONCLUSION

From the tables 4 and 5 it is clear that the difference in analytical and ANSYS Software results is within acceptable limits. This difference is due to numerical techniques of Finite Element Method in ANSYS. Since analytical results are validated by FEM calculations, the design methodology proposed in this paper can be successfully applied into the real-world mechanical applications for minimizing the material volume of multi-layered compound cylinders to assure best utilization of material.

Patil S. A. [2-3] has found minimum volume of two layer compound cylinders as **37974.94** mm^3 (for internal diameter = 100 mm and steel material with yield strength = 250 MPa). Miraje Ayub A. and Patil Sunil A. [5] have found minimum volume of three-layer open type compound cylinder as **31.778.98** mm^3 considering plane stress hypothesis. In comparison with this, plane strain hypothesis gave minimum volume as **30920.07** mm^3 (as per combination set 2) which is quite significant to save the material. Accordingly optimum thickness of cylinder 1 is **28.5** mm, optimum thickness of cylinder

2 is **39.2** mm and optimum thickness of cylinder is **54.5** mm. Hence it can be concluded that plane strain hypothesis gives better results to find optimum thicknesses of long hollow compound cylinders.

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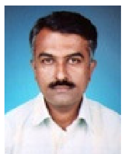

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