WAVE PROPAGATION IN A HOMOGENEOUS ISOTROPIC FINITE THERMO-ELASTIC CIRCULAR PLATE

K. Kadambavanam[1] and L. Anitha[2]
[1]Department of Mathematics, Sri Vasavi College, Erode
[2]Department of Mathematics, Nandha Arts & Science College, Erode

ABSTRACT
In this paper the wave propagation in a homogeneous isotropic circular plate is investigated in the context of the linear theory of elasticity. The fundamental equations are simplified and the free vibration solution for simply supported isotropic circular plate is obtained by using Bessel functions with complex arguments. To clarify the correctness and effectiveness of the developed method, the dispersion curves for length to outer radius ratio are computed and presented for zinc material with the support of MATLAB.

KEYWORDS*: wave propagation, isotropic circular plate, Bessel function.*

I. INTRODUCTION

The analysis of thermally induced vibration of circular plate is common place in the design of structures, atomic reactors, steam turbines, supersonic aircraft, and other devices operating at elevated temperature. In the field of nondestructive evaluation, laser-generated waves have attracted great attention owing to their potential application to noncontact and nondestructive evaluation of sheet materials. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses, reducing the strength of the aircraft structure. In the nuclear field, the extremely high temperatures and temperature gradients originating inside nuclear reactors influence their design and operations. Moreover, it is well recognized that the investigation of the thermal effects on elastic wave propagation has bearing on many seismological application.
The theory of thermo elasticity is presently studied by Nowacki [1]. Lord and Shulman [2] and Green and Lindsay [3] modified the Fourier law and constitutive relations, so as to get hyperbolic equation for heat conduction by taking into account the time needed for acceleration of heat flow and relaxation of stresses. A special feature of the Green–Lindsay model is that it does not violate the classical Fourier's heat conduction law. Vibration of functionally graded multilayered orthotropic circular plate under thermo mechanical load was analyzed by X.Wang et.al [4]Hallam and Ollerton [5].The torsional vibration of piezoelectric solid bone of finite length was investigated by Paul[6].

II. FORMULATION OF THE PROBLEM

Consider a homogeneous isotropic, thermally conducting circular plate of length L having inner and outer radii a & b respectively with central angle α at uniform temperature T_0 in the undisturbed state initially. The governing field equations of motion, strain displacement relation and heat conduction in the absence of body force for a linearly elastic medium are as follows:

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho\,\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\upsilon\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{1}{r}(\sigma_{r\theta}) = \rho\,\frac{\partial^2 v}{\partial t^2} \qquad (1)$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta z} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{1}{r}(\sigma_{rz}) = \rho\,\frac{\partial^2 \omega}{\partial t^2}$$

The heat conduction equation is

$$k\left(\frac{\partial^2 T}{dt^2} + \frac{l}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho C_\gamma \frac{\partial T}{\partial t} + \rho C_\gamma \frac{\partial^2 T}{dt^2} +$$

$$\beta T_o\left[\frac{\partial^2 u}{\partial r\,\partial t} + \frac{1}{r}\left(\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial \theta\,\partial t}\right) + \frac{\partial^2 \omega}{\partial t \partial z}\right]$$

Stress - Strain relation for an isotropic material by generalized Hooke's law is given by,

$$\sigma_{rr} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{rr} - \beta T$$

$$\sigma_{\theta\theta} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{\theta\theta} - \beta T$$

$$\sigma_{zz} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{zz} - \beta T \qquad (2)$$

$$\sigma_{r\theta} = \mu\gamma_{r\theta}\ ;\ \sigma_{\theta z} = \mu\gamma_{\theta z};\ \sigma_{rz} = \mu\gamma_{rz}$$

Where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ are the normal stress components and $\sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr}$ are the shear stress components and ρ is the mass density of the circular plate.

The strain - displacement relation is given by,

$$e_{rr} = \frac{\partial u}{\partial r};\quad e_{\theta\theta} = \frac{\mu}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta};\quad e_{zz} = \frac{\partial \omega}{\partial z};$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta};\quad \gamma_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial \omega}{\partial \theta};\quad \gamma_{rz} = \frac{\partial \omega}{\partial r} + \frac{\partial u}{\partial z} \qquad (3)$$

Here u, v, w are the displacement vectors along radial, circumferential and axial directions respectively. T(r,θ,z,t) is the temperature change, λ & $\mu$are the elastic constants. C_v is the specific heat capacity,e_{rr}, $e_{\theta\theta}$, e_{zz} are the normal strain components and $e_{r\theta}$, $e_{\theta z}$, e_{zr} are the shear strain components.

Substituting the equations (3)&(2) in equation (1) gives the following displacements equation of motion,

$$(\lambda + 2\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + \frac{\mu}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \mu\frac{\partial^2 u}{\partial z^2} +$$

$$\left(\frac{\lambda + \mu}{r}\right)\frac{\partial^2 v}{\partial r\,\partial \theta} - \left(\frac{\lambda + 3\mu}{r^2}\right)\frac{\partial v}{\partial \theta} + (\lambda + \mu)\frac{\partial^2 \omega}{\partial r\,\partial z} - \beta\frac{\partial T}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{\lambda+\mu}{r}\right)\frac{\partial^2 u}{\partial r\,\partial\theta}+\left(\frac{\lambda+3\mu}{r^2}\right)\frac{\partial u}{\partial\theta}+\mu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)+\left(\frac{\lambda+2\mu}{r^2}\right)\frac{\partial^2 v}{\partial\theta^2}$$

$$+\mu\frac{\partial^2 v}{\partial z^2}+\left(\frac{\lambda+\mu}{r}\right)\frac{\partial^2\omega}{\partial\theta\,\partial z}-\beta\frac{\partial T}{\partial\theta}=\rho\frac{\partial^2 v}{\partial t^2}$$

$$(\lambda+\mu)\frac{\partial^2 u}{\partial r\partial z}+\left(\frac{\lambda+\mu}{r}\right)\frac{\partial u}{\partial z}+\left(\frac{\lambda+\mu}{r}\right)\frac{\partial^2 v}{\partial\theta\,\partial z}+\mu\left[\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}+\frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2}\right]+$$

$$(\lambda+2\mu)\frac{\partial^2\omega}{\partial z^2}-\beta\frac{\partial T}{\partial z}=\rho\frac{\partial^2\omega}{dt^2}$$

$$\rho C_\gamma k\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}+\frac{\partial^2 T}{\partial z^2}\right]-\rho C_\gamma\frac{\partial^2 T}{\partial t^2}-\rho C_\gamma\frac{\partial T}{\partial t}$$

$$-\beta T_o\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial\omega}{\partial z}\right]=0 \quad (4)$$

The equation (4) is a coupled partial differential equation of three displacement components. To uncouple the equation (4), consider the following solution of (4) as follows:

$$u=\frac{1}{r}\psi,_{\theta}-\phi,_{r}\;;\quad v=-\frac{1}{r}\phi,_{\theta}-\psi,_{r}\;;\quad w=-\psi,_{z} \quad (5)$$

Using equation(5) in equation (4) yields the following second order partial differential equation with constant coefficients.

$$\therefore\left[(\lambda+2\mu)\nabla_1^2+\mu\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right]\phi+(\lambda+\mu)\frac{\partial^2\psi}{\partial z^2}$$

$$+\beta T+\mu\left[\nabla_1^2+\frac{\partial^2}{\partial z^2}-\frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right]\psi=0$$

$$\left[\left(\lambda+2\mu\right)\nabla_1^2+\mu\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right]\phi+(\lambda+\mu)\frac{\partial^2\psi}{\partial z^2}$$

$$-\beta T+\mu\left[\nabla_1^2+\frac{\partial^2}{\partial z^2}-\frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right]\psi=0 \quad (6)$$

$$\left[\left(\mu\nabla_1^2+(\lambda+2\mu)\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\psi+(\lambda+\mu)\nabla_1^2\phi+\beta T\right]=0$$

$$\nabla_1^2+\frac{\partial^2 T}{\partial z^2}-\frac{\tau}{C_\gamma k}\frac{\partial^2 T}{\partial t^2}-\frac{1}{k}\frac{\partial T}{\partial t}-\frac{\beta T_o(i\omega)}{\rho C_\gamma k}\left[\nabla_1^2\phi+\frac{\partial^2\psi}{\partial z^2}\right]=0 \quad (7)$$

In case of torsional vibration, the only non-vanishing mechanical displacement V(r, z, t) along the cross-radial direction and the thermal potential T(r, z, t) are independent of θ.

III. SOLUTION TO THE PROBLEM

The equation (7) is coupled partial differential equations of the three displacement components. To uncouple equation (7), we can write three displacement functions which satisfies the simply supported boundary conditions.

The displacement function and temperature change is given by,

$$\phi = \bar{\phi}(r)\ \sin(m\pi z)\ \sin\left(\frac{n\pi}{\alpha}\right)\theta\ e^{i\omega t}$$
$$T = \bar{T}(r)\ \sin(m\pi z)\ \sin\left(\frac{n\pi}{\alpha}\right)\theta\ e^{i\omega t} \tag{8}$$

Where *m* is the circumferential mode and *n* is the axial mode, ω is the angular frequency of the circular plate. Introducing the non-dimensional quantities,

$$r' = \frac{r}{R};\quad z' = \frac{z}{L};\ \bar{T} = \frac{T}{T_o};\ \Omega^2 = \frac{\rho\omega^2 R^2}{\lambda + 2\mu};\ \in_1 = \frac{T_o R}{\rho^2 C\gamma C_1 k}\beta^2;\ \in_2 = \frac{TC_1^2}{C_\gamma k}$$

$$\in_3 = \frac{C_1 R}{k};\ \delta = \frac{n\pi}{\alpha};\ \bar{\lambda} = \frac{\lambda}{\mu};\ \in_4 = \frac{1}{2+\bar{\lambda}};\ C_1^2 = \frac{\rho}{\lambda + 2\mu};\ \alpha' = \frac{\delta}{R^2}$$

Substituting the equation (8) in equation (7), yields the following second order partial differential equation,

Since $\nabla_2^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial^2},$

$$(\nabla_2^2 + \Omega^2)\bar{\phi} + \frac{\beta T_0 R^2}{\lambda + 2\mu}\bar{T} = 0$$
$$(\nabla_2^2 + \in_2 \Omega^2)\bar{T} + \in_1 \Omega \nabla_2^2 \bar{\phi} = 0$$
$$(\nabla_2^2 + g_1)\bar{\phi} + g_2 \bar{T} = 0$$
$$(\nabla_2^2 + \in_2 \Omega^2)\bar{T} + g_3 \nabla_2^2 \bar{\phi} = 0 \tag{9}$$

where $g_1 = \Omega^2$

$g_2 = \frac{\beta T_0 R^2}{\lambda + 2\mu}$

$g_3 = \in_1 \Omega$

The system of equations given in equation (9) has trivial solution. To obtain the non-trivial solutions, the coefficient of the determinant is equal to zero, that is

$$\begin{vmatrix} \nabla_2^2 + g_1 & g_2 \\ g_5 \nabla_2^2 & \nabla_2^2 + \in_2 \Omega^2 \end{vmatrix} (\bar{\phi}, \bar{T}) = 0 \tag{10}$$

On simplification of (10), the equation becomes,

$$\nabla_2^4 + A\nabla_2^2 + B = 0 \quad (11)$$

where $A = g_1 + \in_2 \Omega^2 - g_2 g_5$ & $B = g_1 \in_2 \Omega^2$

The solution of equation (11)is

$$\bar{\phi}=\sum_{i=1}^{2} A_i J_\delta(\alpha_{ir})\sin(m\pi z)\ \sin\left(\frac{n\pi}{2}\right)\theta$$
$$\bar{T}=\sum_{i=1}^{2} \rho_i A_i J_\delta(\alpha_{ir})\sin(m\pi z)\ \sin\left(\frac{n\pi}{2}\right)\theta \qquad (12)$$

J_δ is the Bessel function of the first kind.

IV. Boundary Conditions And Frequency Equations

$\sigma_{r\theta}=0=V$ at r = a and

$\sigma_{rz}=0=T$ at Z = ± L

$T_{,r}=0$ (13)

Using the result obtained in the equations (1)-(3) in equation (13) we can get the frequency equation of uncoupled free vibration as follows:

The frequency equation is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=0 \qquad (14)$$

$a_{11}=(2+\lambda)([\delta J\delta(\propto_1 t_1)/t_1^2-\frac{\propto 1}{t1}J\delta+1(\propto_1 t_1)])-(\propto_1 t_1)^2R^2-\delta^2)J\delta(\propto_1 t_1)/t_1^2)$
$\quad +\lambda(\delta(\delta-1)J\delta(\propto_1 t_1)/t_1^2-\frac{\propto 1}{t1}J\delta+1(\propto_1 t_1))+\lambda d_1 t_L^2 J\delta(\propto_1 t_1)$

$a_{13}=(2+\lambda)[(\delta J\delta(\propto_2 t_1)/t_1^2-\frac{\propto 2}{t1}J\delta+1(\propto_2 t_1)])-(\propto_2 t_1)^2R^2-\delta^2)J\delta(\propto_2 t_1)/t_1^2)$
$\quad +\lambda(\delta(\delta-1)J\delta(\propto_2 t_1)/t_1^2-\frac{\propto 2}{t1}J\delta+1(\propto_2 t_1))+\lambda d_1 t_L^2 J\delta(\propto_2 t_1)$

$a_{15}=(2+\lambda)[\frac{K1\delta}{t1}J\delta+1(K_1 t_1)-\delta(\delta-1)J\delta(K_1 t_1)/t_1^2]+\lambda[\delta(\delta-1)J\delta(K_1 t_1)/t_1^2$
$\quad -\frac{K1\delta}{t1}J\delta+1(K_1 t_1)]$

$a_{21}=2\delta[(\frac{\propto 1}{t1})J\delta+1(\propto_1 t_1)-\delta(\delta-1)J\delta(\propto_1 t_1)]$

$a_{23}=2\delta[(\frac{\propto 2}{t1})J\delta+1(\propto_2 t_1)-\delta(\delta-1)J\delta(\propto_2 t_1)]$

$a_{25}=(K_1 t_1)^2R^2 J\delta(K_1 t_1)-\frac{2\delta(\delta-1)}{t12}J\delta(K_1 t_1)+\frac{k1}{t1}J\delta+1(K_1 t_1)$

$a_{31}=-t_L(1+d_1)(\frac{\delta}{t1}J\delta(\propto_1 t_1)-\propto_1 J\delta+1(\propto_1 t_1))$

$a_{33}=-t_L(1+d_2)(\frac{\delta}{t1}J\delta(\propto_2 t_1)-\propto_2 J\delta+1(\propto_2 t_1))$

$a_{35}=-t_L(\frac{\delta}{t1})J\delta(K_1 t_1))$

In which $t_1=\frac{a}{R}=1-\frac{t*}{2}$, $t_2=\frac{b}{R}=1+\frac{t*}{2}$ and $t^*=\frac{b-a}{R}$ is the thickness to mean radius ratio of the plate. Obviously, a_{ij} (j = 2,4,6) can obtained by just replacing the modified Bessel function of the first kind in a_{ij} (i= 1,3,5) with the ones of the second kind, respectively, while a_{ij} (i= 4,5,6) can be obtained by just replacing t_1 in a_{ij} (j= 1,2,3) with t_2.

V. Numerical Results And Discussion

The frequency equation (14) is numerically solved for Zinc material. For the purpose of numerical computation, consider the closed circular plate with the center angle $\alpha=2\pi$ and the integer n must be even, since the shell vibrates in circumferential full wave. The material properties of a Zinc are

$\rho=7.14\times10^3 kgm^{-3}$, T_0=296 K

$\mu = 0.508 \times 10^{11} Nm^{-2}$, β=1

$\lambda = 0.385 \times 10^{11} Nm^{-2}$ and Poisson ratio ν=0.3. C_v =3.9×10^{2} J kg^{-1} deg^{-1}

The roots of the algebraic equation (11) were calculated using a combination of Newton-Raphson method. A dispersion curve is drawn between the non-dimensional circumferential wave number versus dimensionless frequency for the different thickness parameters t*= 0.1, 0.25, 0.5 with the axial wave number in first and second mode is shown in Figure.1 and Figure.2 respectively.

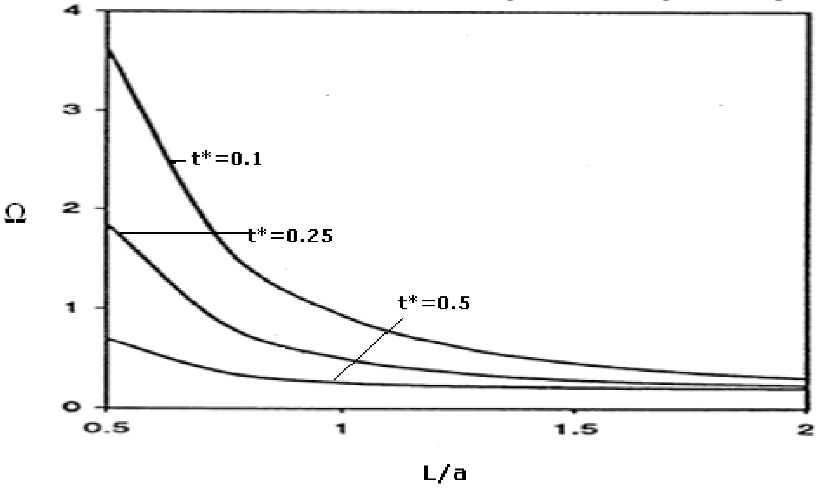

Figure.1.Variation of frequency versus L/a in first mode

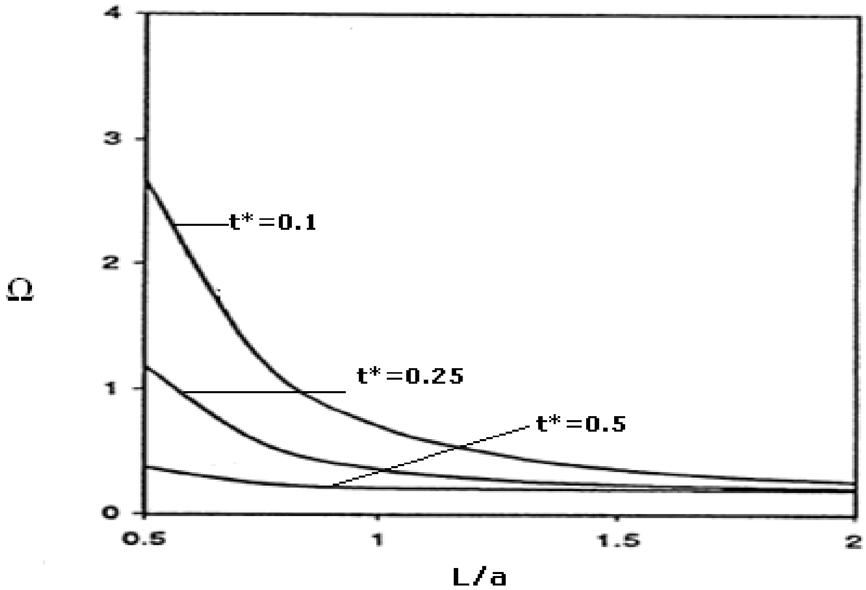

Figure.2. Variation of frequency versus L/a in first mode

From the Figures.1 and 2, it is observed that the non-dimensional frequency decreases rapidly to become linear at L/a=1 for both first and second mode. When the thickness of the circular plate is increased, the dimensionless frequency is decreased. This is the proper physical behavior of a circular plate with respect to its thickness. The comparison of Figure.1 and Figure.2 shows that the non-dimensional frequency decrease exponentially for L/a<1 in the two modes, but the case when L/a>1 the non-dimensional frequency is steady and slow for all values of t*.

By comparing with the classical thin shell theory (CTST), it is clear that the exact one agree well with increase in thickness to mean radius ratio. This is identical to the well-known property of CTST for the uncoupled problem. However, for the thinner panel, when the effect of the foundation is obvious, the frequency of CTST will become smaller than the exact one.

VI. CONCLUSION

In this paper, the wave propagation of a homogeneous isotropic finite thermo-elastic circular plate is analyzed by satisfying the boundary conditions using Bessel functions with complex arguments. Numerically the frequency equations are analyzed for zinc material. The computed non-dimensional wave numbers are presented in the form of dispersion curves. The method proposed in this paper can be used to analyze the torsional vibration of a finite thermo-elastic circular plate.

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AUTHORS BIOGRAPHY

K. Kadambavanam was born in Palani, Tamilnadu, India, in 1956. He did his post-graduate studies at Annamalai University in 1979. He completed his M.Phil, degree in Annamalai University in 1981. He obtained his Ph.D., degree from Bharathiar University, Coimbatore. His area of research is Fuzzy Queueing Models and Fuzzy Inventory Models. His research interests include Solid mechanics, Random Polynomials and Fuzzy Clustering. He has 33 years of teaching experience in both U.G and P.G level and 18 years of research experience. He is a Chairman of Board of Studies (P.G) in Mathematics and Ex-Officio Member in Board of Studies (U.G) in Mathematics in Bharathiar University, Coimbatore. He is a member in : Panel of Resource Persons, Annamalai University, Annamalai Nagar, Doctorial Committee for Ph.D., program, Gandhigram Rural University, Gadhigram'.

L.Anitha was born in Erode, Tamil Nadu, India, in 1978. She did her post-graduate studies at Madurai Kamaraj University in 2000. She completed her M.Phil., degree in Bharathiar University. She is doing her Ph.D., degree in Sri Vasavi College, Erode which is affiliated to the Bharathiar University, Coimbatore. Her area of research is Vibrations in Solid Mechanics. She has 10 years of teaching experience in both U.G and P.G level and 4 years of research experience through guiding the M.Phil., Students. She is the Question paper Setter and Examiner of the various Universities and Autonomous Colleges'.