EXPERIMENTAL AND COMPUTATIONAL VALIDATION OF AN ANALYTICAL MODEL OF FREE VIBRATION OF A RECTANGULAR PLATE CARRYING A DISTRIBUTED MASS

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ABSTRACT

This paper is concerned with the experimental and computational validation of a mathematical approach to the free vibration of a rectangular plate carrying a distributed mass, which is originally developed by Kopmaz and Telli in [Journal of Sound and Vibration, 251(1) (2002) 39-57]. In the experiments, an aluminum plate is simply supported by many hinges and two different types of distributed masses are used. Each attachment is placed on the center and on the center of quarter of the plate, and the eigenfrequencies are obtained for these four cases. Furthermore, the structural system is modelled in the computational environment by using finite element method in order to compare the natural frequencies. It is observed that the theoretical, experimental and FE results of the modal analysis are quite close to each other. For further analysis, the modal shapes of the system under one of the specified cases that the attachment is placed on the center of the plate, obtained from the theoretical model and FE analysis are also represented and the results are very close to each other. Thus, the validity of the analytical model of the free vibration of a constrained plate is proved for the first time in this paper.

KEYWORD: *free vibration, constrained system, rectangular plate, distributed mass*

I. INTRODUCTION

Most structural elements used in engineering systems often differ from those which are considered in the textbooks on the strength of materials or the theory of elasticity in their irregular shaped, complex boundary conditions, and additional constrains. Here, the word ‘constrains’ imply that these structural elements are equipped with some concentrated or distributed masses or be connected with some flexible, spring-like elements beside their boundaries, hence, they cannot move or vibrate freely as bare plates, beams etc. do. Therefore, their equations of motion will be subject to some modification compared to those without such constrains, as expected. In today’s mechanical systems one frequently encounters such constrained elements, in order words structural elements with attachments. As a consequence of this fact, in the last two decades, intensive research has been conducted in this field. This literature developed in two main categories, i.e., beams and plates with attachment(s). Since the present paper focuses on the experimental verification of the natural vibration of a plate carrying a distributed mass, the literature related to constrained plates will be confined.

Wu and Lou [1] developed an analytical and numerical combined method for the free vibration analysis of rectangular plates with any number of lumped masses and translational springs. In this method, one starts with the forced vibration equation of a plate, and using assumed modes method and the orthogonality of comparison functions, a set of ordinary differential equations. Afterwards, forcing terms on the right hand side of these equations are replaced with the terms reflecting the dynamics effect of attachment. Assuming that this combined system performs a harmonic motion, one arrives at an eigenvalues problem. This problem is solved using a computer. In [2], Cha dealt with the free vibration of a rectangular plate carrying a concentrated mass, and compared the two methods given in

[1], and showed that the hybrid method that the author himself developed yields the same frequencies as the exact method does, and the method in [1] gives lower values than the exact ones. Cha and Wong presented a novel approach in [3] to determine the natural frequencies of a constrained system -plate or beam-. This method is based on the assumed modes method. When the real system are discretized by N modes, the eigenvalues of an NxN matric, the authors employed a methodology proposed by Golub, and reduced the problem to find the eigenvalues of an SxS matrix, where S is the number of attachment points, and S<N. Low presented an improved model to estimate the natural frequencies of mass-loaded plates [4]. He introduced the change in strain energy owing to the mass difference into his model, defining a new factor called stiffness factor. Essential to the model is the concept of equivalent center weight factor method proposed by the author in his a previous paper.

Kopmaz and Telli [5] developed an effective method to analyze free vibration of a rectangular plate carrying a distributed mass. In their work, it is assumed that a distributed mass of rectangular shape is located on somewhere of the plate, being its edges parallel to the edges of the plate. To establish a mathematical model considering the dynamic effect of distributed mass, they employed the Heaviside unit step functions. The proposed model allows one to find natural frequencies of a plate carrying a concentrated mass as a limited case. The equation of motion of the plate is discretized by means of the Galerkin's procedure. For different locations of the distributed mass, the natural frequencies and modal surfaces are obtained. The results obviously show how important a distributed mass affects free vibration properties of a plate. Wong proposed an approach to the problem of finding natural frequencies of a plate carrying a distributed mass based on Rayleigh-Ritz method [6]. Similar to that in [5], he obtained modal shapes of the constrained plate. Wu et al. [7] established a mathematical model for free vibration of a plate carrying multiple various concentrated masses via linear springs. They used the Delta-Dirac functions to represent the pointwise connected attachments for various boundary conditions. Amabili et al. [8] studied the effect of rotary inertia of a concentrated mass on vibrations of a rectangular plate. They showed mathematically and experimentally that the rotary inertia of a mass reduces natural frequencies of a combined system and changes the shape of its modal forms. Moreover, it causes new additional modes to appear. The authors also consider the imperfect geometry of the plate. Wu, in [9], proposed a different technique to analyze free vibrations of a plate to which a distributed mass is attached by linear springs. This attachment adds 3 degrees of the freedom to the infinite DOF of the plate. The author replaces this spring-supported mass with equivalent point masses. Zu and Ji [10] investigated the free vibration of rectangular plates supporting a continuously distributed mass by continuously distributed linear springs. As special cases, they studied free vibrations of rectangular plates partially occupied by uniformly distributed springs-mass. Malekzadeh et al. [11] treated free vibrations of thick rectangular composite plate that carries an uniformly distributed mass including stiffness effect. The authors used the higher order plate theory. To introduce the distributed mass in the model they employed the method developed developed by Kopmaz [5]. In the paper, the significance of wring a higher order plate theory for thick plate is emphasized. In [12], Zhou and Ji presented a direct method to derive the exact solution for the free vibration of thin rectangular plates with discrete spring masses. Two opposite edges of the plate are simply supported while the remaining two are elastically restrained. They obtained exact expressions for mode and frequency equations of plates with attachments. Amabili and Carra dealt with large amplitude vibrations of rectangular plates carrying concentrated masses. In the experiments, they put the plate in two different position, i.e., vertically and horizontally, [13]. In the vertical position, the hardening effect in the frequency response graphics due to large amplitude vibration is negligible because of the absence of gravity deflection while in the horizontal position, the attached mass increases the deflection nonlinearly resulting in an obviously observable hardening stiffness. In the practice, adding a concentrated or distributed mass can be used for the purpose of attenuating forced vibration response of a plate. Mahadevaswamy and Suresh [14] tried to find the optimal mass ratio of vibratory flap for suppressing forced vibrations of a clamped plate. To this end, they performed both finite element modelling and experiments. Salleh et al. [16] studied the effects of the absorber attachment on the vibration response of simply supported rectangular plate. It was shown that the right position and number of the absorber can be the main contribute to suppress the vibration on the plate.

To the authors' knowledge, the use of Heaviside functions to define the distributed masses on a freely vibrating plate was made by Kopmaz and Telli [5] first time. In that paper, a plate carrying a distributed mass of rectangular shape was considered. This paper is devoted to the experimental and computational verification of the validity of the proposed method. For this purpose, a rectangular plate simply supported by small hinges was used in the experiments. For different positions of distributed mass, the plate was excited by a modal shaker and acceleration data were acquired by means of a piezoelectric sensor. Furthermore, the plate with distributed mass was modeled in computer environment using finite element method. Natural frequencies of the constrained plate were compared with those obtained experimentally and from the simulations. The eigenfrequencies obtained from the theoretical model which was originally published by Kopmaz and Telli in [5] were observed to be quite close to the experimental ones within an order of error 5-10 % and much closer to the FE results. Furthermore, the modal shapes of the system obtained from the theory and the FE model are also given for one of the cases treated here. The possible reason of the small mismatching between the results is considered to be the effect of the rotary inertia of the mass and the imperfections of the boundary conditions. However, the both the experimental measurement and the FE results show that the theoretical modal in [5] works out well.

II. MODELING OF THE PLATE WITH ATTACHMENT

In this part, the theoretical model developed by Kopmaz and Telli [5] is briefly reviewed. Afterwards, the finite element model will be explained. The rectangular plate under study is shown in the Figure 1. The width, length and thickness of the plate are a, b and h, respectively. The assumptions are that the plate is isotropic, simply supported along all its edges and its thickness does not vary and the rotary inertia effect of the distributed mass is neglected. The dimensions of the attachment c and d, and its coordinates are shown in Figure 1.

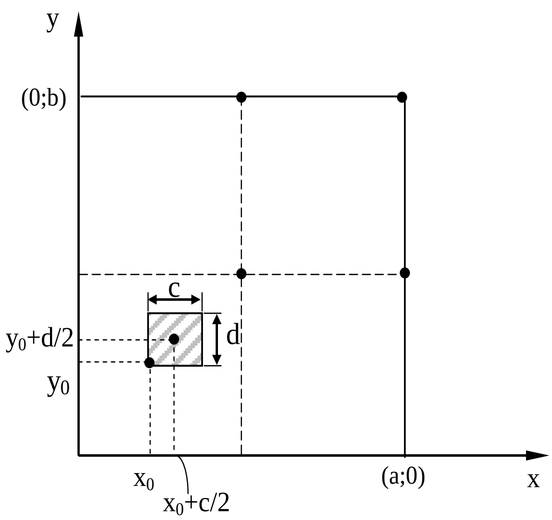

Figure 1. Schema of the rectangular plate with distributed attachment

The equation of motion of the plate carrying a distributed mass is obtained as follows:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\bar{\rho}\frac{\partial^2 w}{\partial t^2}=-\rho\frac{\partial^2 w}{\partial t^2}\mathcal{H}(x,y,x_0,y_0,c,d) \tag{1}$$

where $w=w(x,y,t)$ is the transversal deflection of the plate, $\bar{\rho}$ and ρ are the area density of the plate and the distributed mass, respectively. D is the flexural rigidity of the plate and is defined as follows:

$$D=\frac{E\,h^3}{12(1-\nu^2)} \tag{2}$$

where E, ν and h are Young modulus, Poisson's ratio and the plate thickness, respectively. In Equation (1), the term $\mathcal{H}$ is a combination of four Heaviside functions as described below:

$$\mathcal{H}(x,y,x_0,y_0,c,d) = \mathcal{H}_1(x-x_0,y-y_0) - \mathcal{H}_2(x-x_0,y-y_0-d) - \mathcal{H}_3(x-x_0-c,y-y_0) + \mathcal{H}_4(x-x_0-c,y-y_0-d) \tag{3}$$

The functions $\mathcal{H}_i$'s have the following explicit forms:

$$\begin{aligned} \mathcal{H}_1(x-x_0,y-y_0) &= H(x-x_0)\cdot H(y-y_0) \\ \mathcal{H}_2(x-x_0,y-y_0-d) &= H(x-x_0)\cdot H(y-y_0-d) \\ \mathcal{H}_3(x-x_0-c,y-y_0) &= H(x-x_0-c)\cdot H(y-y_0) \\ \mathcal{H}_4(x-x_0-c,y-y_0-d) &= H(x-x_0-c)\cdot H(y-y_0-d) \end{aligned} \tag{4}$$

Using the Galerkin discretization procedure for Equation (1), it is approached to the true solution of this equation by using the following series:

$$w(x,y,t) = \sum_{i=1}^{Nx}\sum_{j=1}^{Ny} W_{ij}(x,y)\,q_{ij}(t) \tag{5}$$

where W_{ij} s are the modal functions of the simply supported bare plate, and they are given below:

$$W_{ij}(x,y) = \sin\frac{i\pi x}{a}\ \sin\frac{j\pi y}{b} \tag{6}$$

If we substitute equation (5) in equation (1), multiply each term by $W_{rs}(x,y)$, and integrate all terms in the equation over the domain ($0\le x\le a$, $0\le y\le b$) we have:

$$\pi^4\left[\left(\frac{r}{a}\right)^2+\left(\frac{s}{b}\right)^2\right]^2 D\frac{ab}{4}q_{rs} + \bar{\rho}\frac{ab}{4}\ddot{q}_{rs} + \rho ab\left[\sum_i^{Nx}\sum_j^{Ny}\bar{I}_{ij,rs}\,\ddot{q}_{ij}\right]=0, \quad r=1,...,N_x; s=1,...,N_y$$

(7)

Substituting $\ddot{q}_n = -\omega^2 q_n$ (8)

in the set of equation (7) yields

$$\bar{\omega}_m^2 q_m - \sum_{n=1}^{N}\left[4\left(\frac{\rho}{\bar{\rho}}\right)\bar{I}_{n,m}+\delta_{nm}\right]\omega^2 q_n = 0 \quad m=1,...,N \quad (N=N_xN_y) \tag{9}$$

or in the matrix form

$$\left\{\omega^2\left\{[I]+[Q]\right\} - \left[\bar{\omega}^2\right]\right\}\{q\} = \{0\} \tag{10}$$

In order that Equation (10) has non-trivial solutions, the determinant of the coefficients' matrix of the vector {q} must be equal to zero. This requirement leads to an polynomial of degree n in ω, the roots of which are the natural frequencies of the system. For further details, see [5].

In order to verify the theoretical model, the rectangular plate carrying a distributed mass is modelled using by finite element method in the computational environment. The plate is defined by shell elements and the attachment is considered as a solid block. The plate is divided into 2396 finite elements and the number of nodes is 2727.

III. EXPERIMENTAL SETUPS

The general view of the experimental test rig is demonstrated in Figure 2a. The aluminum rectangular plate shown in Figure 2b is simply supported, and its dimensions and material properties are given in Table 1. The plate is excited by a modal shaker with maximum force capacity of 100 N. The response of the plate is measured with a very small and light piezoelectric accelerometer. In order to satisfy the ideal boundary conditions, small hinges of sufficient number are used.

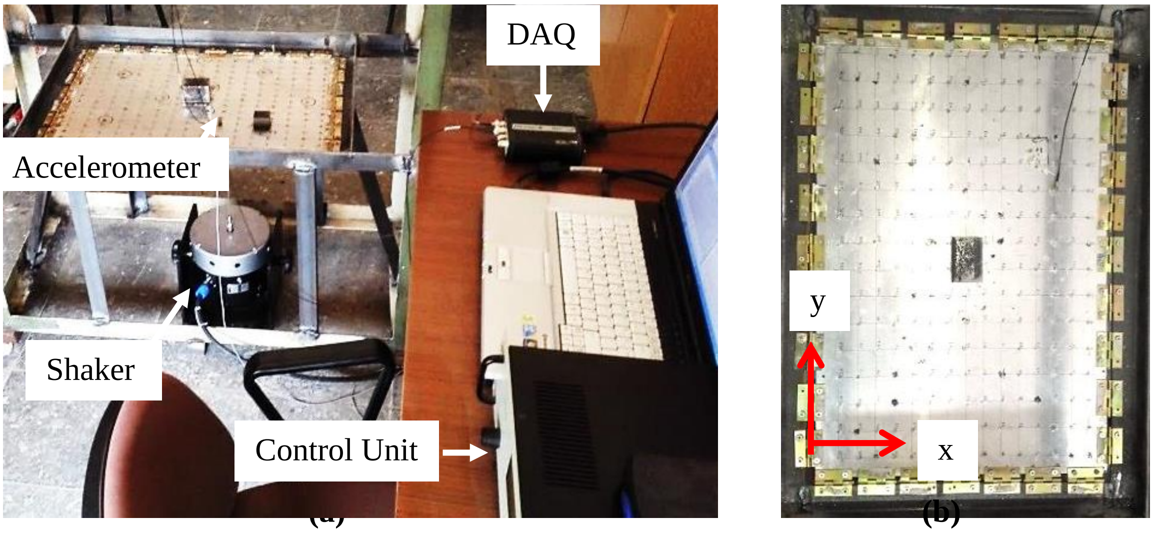

Figure 2. Experimental Test rig: (a) general overview, (b) well-nigh simply supported rectangular plate

In order to determine the natural frequency of the plate, a sine sweep in the range of 0-200 Hz is performed by the modal shaker. Modal analysis is carried out using a special software and the acceleration data are collected from 240 different measurement points on the plate.

Table 1. Dimensions and material properties of the rectangular plate used in experiments

Dimensions of Plate		Material Properties of Plate	
Length	590 mm	Elasticity modulus	70 GPa
Width	400 mm	Poison's ratio	0,35
Thickness	2 mm	Density	2630 kg/m^3

IV. RESULTS AND DISCUSSION

The theoretical, experimental and FE results of the free vibration analysis of the plate are presented here. Firstly, the experimental eigenfrequencies of simply supported bare plate are determined to check the validity of the experimental conditions since the natural frequencies of such a plate are available in the literature. After that, the natural frequencies of the rectangular plate carrying distributed masses obtained from the theoretical model, experiments and FE analysis are given in the tables. In these analyses, two different attachments and two different positions of them are considered, and so four different cases are studied. Finally, the modal shapes of the constrained plate are plotted for one of the cases, which are obtained from the theoretical and FE models.

In Figure 3, the experimental result of the averaged sum of the frequency response functions (FRFs) of the simply supported rectangular plate without attachment is demonstrated. It should be noted that the amplitude of the FRFs is referred to the 1 m/s^2/N in all results. In Table 2, the five natural frequencies of the simply supported rectangular plate without attachment, which are obtained from theoretical model, experiments and the finite element method, are listed. It can be seen from Table 2, the results are quite close to each other and it is concluded that the physical boundary conditions are reliable for further analyses.

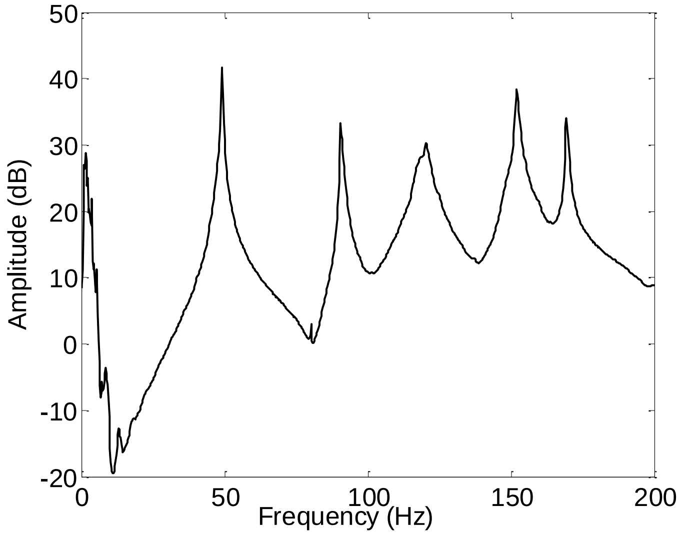

Figure 3. Experimental result of the sum of frequency response functions (FRFs) of the simply supported rectangular plate without attachment

Table 2. The first five frequencies of the simply supported rectangular plate without attachment

Mode	Theoretical Frequency [Hz]	Experiment Results [Hz]	FE Model Solutions [Hz]
ω_1	45,56	48,9	45,62
ω_2	88,61	90,5	88,871
ω_3	139,22	120,3	140,66
ω_4	160,35	152,0	161,81
ω_5	182,27	170,1	183,74

In Figure 4, the sum of frequency response functions (FRFs) of the rectangular plate with the attachment the mass, the length, the width and the height of which are 195,5 gr, 35 mm, 30 mm and 25 mm, respectively, and is located at the center of the quarter of the plate (Case 1) and at the center of the plate (Case 2) are demonstrated. In the Table 3, the five eigenfrequencies obtained from theoretical model, experiments and the finite element method are listed for these two cases. The results are quite close to each other for both two cases as seen from the Table 3.

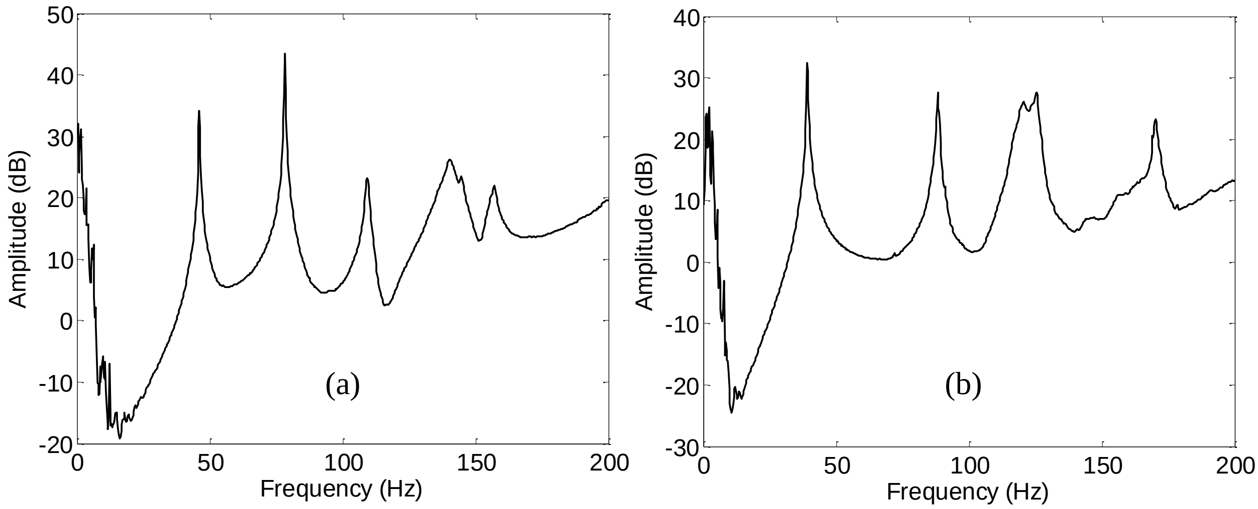

Figure 4. Accelerance in dB versus frequency for the rectangular plate with attachment 1 located at the center of the quarter of the plate (a), the center of the plate (b)

Table 3. First five frequencies of the plate with attachment 1

	CASE 1				CASE 2		
	f_T (Hz)	f_E (Hz)	$f_{FE.}$ (Hz)		f_T (Hz)	f_E (Hz)	$f_{FE.}$ (Hz)
ω_1	41,71	46,5	42,145		35,36	38,9	36,158
ω_2	75,94	78,4	77,946		88,31	88,4	87,693
ω_3	120,79	110,0	121,87		132,90	120,8	136,43
ω_4	152,68	142,9	156,1		138,46	125,7	140,90
ω_5	168,45	157,3	169,75		182,25	170,3	187,21

For further verification of the theoretical model, a larger distributed mass of 720 gr with its dimensions 60 mm×40 mm×40 mm (denoted attachment 2) is used in the experiments and the above mentioned process is repeated.
The experimental results related to the plate with attachment 2 are shown in Figure 5 is located at the center of the quarter of the plate (Figure 5a, Case 3) and at the center of the plate (Figure 5b, Case 4). The first five natural frequency of the structure found from theoretical model, experiments and the finite element method are listed. It can be also concluded that the results are quite close to each other for both two cases as seen from Table 4.

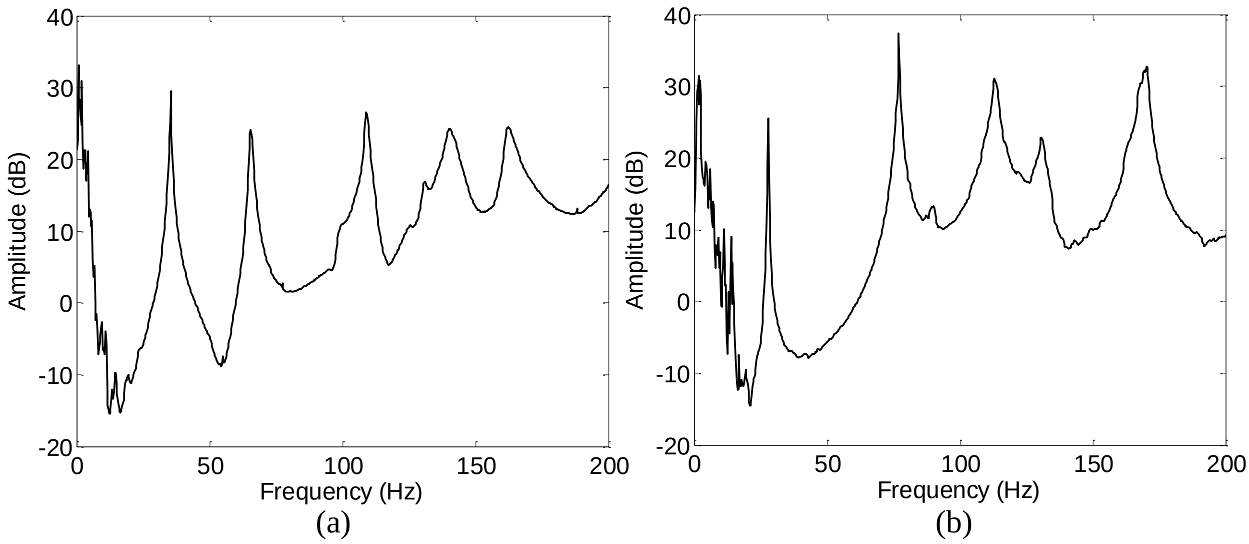

Figure 5. Accelerance in dB versus frequency for the rectangular plate with attachment 2 that located at the quarter of the center (a) and the center of the plate (b)

Table 4. First five frequencies of the plate with attachment 2

	CASE 3				CASE 4		
	f_T (Hz)	f_E (Hz)	$f_{FE.}$ (Hz)		f_T (Hz)	f_E (Hz)	$f_{FE.}$ (Hz)
ω_1	32,72	35,2	34,46		24,35	27,5	26,14
ω_2	64,21	65,5	68,22		83,57	77,3	80,05
ω_3	112,06	109,1	114,63		117,11	113,2	113,99
ω_4	147,34	140,5	138,14		133,62	130,6	140,51
ω_5	165,72	161,7	162,34		180,15	169,4	194,66

The first five modal shapes of the rectangular plate carrying the distributed mass obtained from theoretical model and FE analysis for Case 1 are presented in Figure 6 and Figure 7, respectively. It is observed that the modal shapes of the plate are very similar to each other. It should be noted that third and fourth modal shapes from the theoretical model are interchanged. This is generally caused by a phenomenon called curve veering which is shown to be a result of discretization process [16], but there is some cases in which this interchange between mode functions occur physically [17].

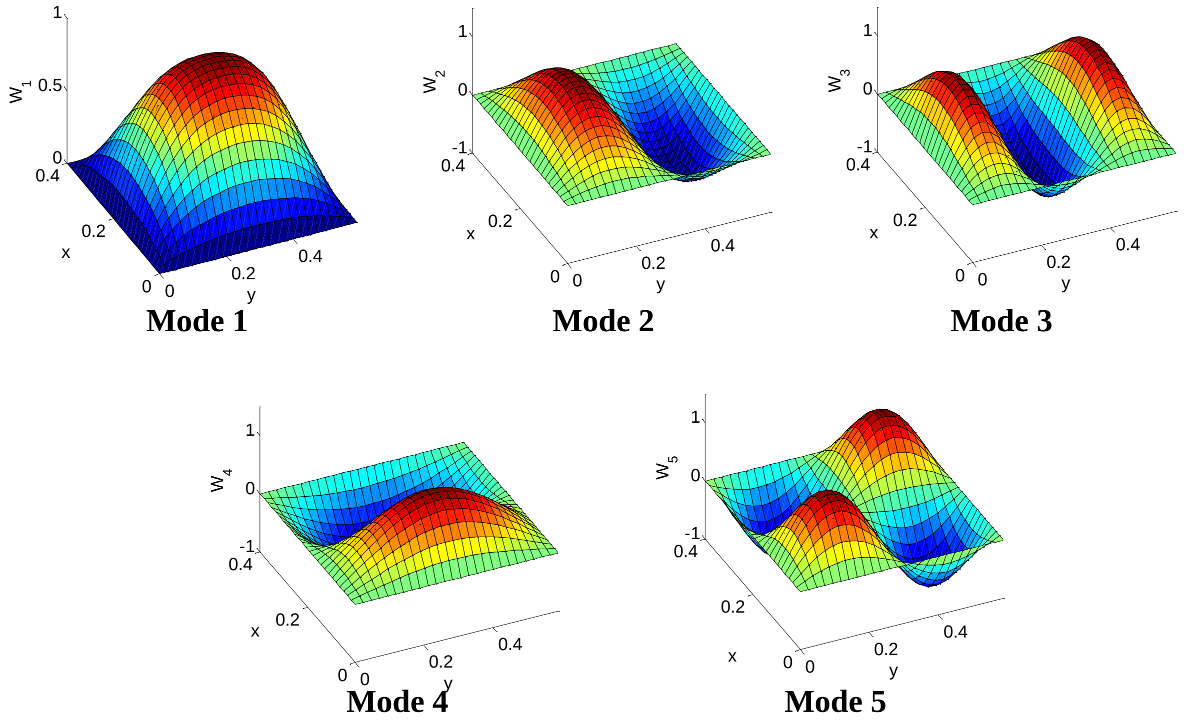

Figure 6. The first five modal shape of the rectangular plate carrying a distributed mass obtained from the theoretical model for the Case 1

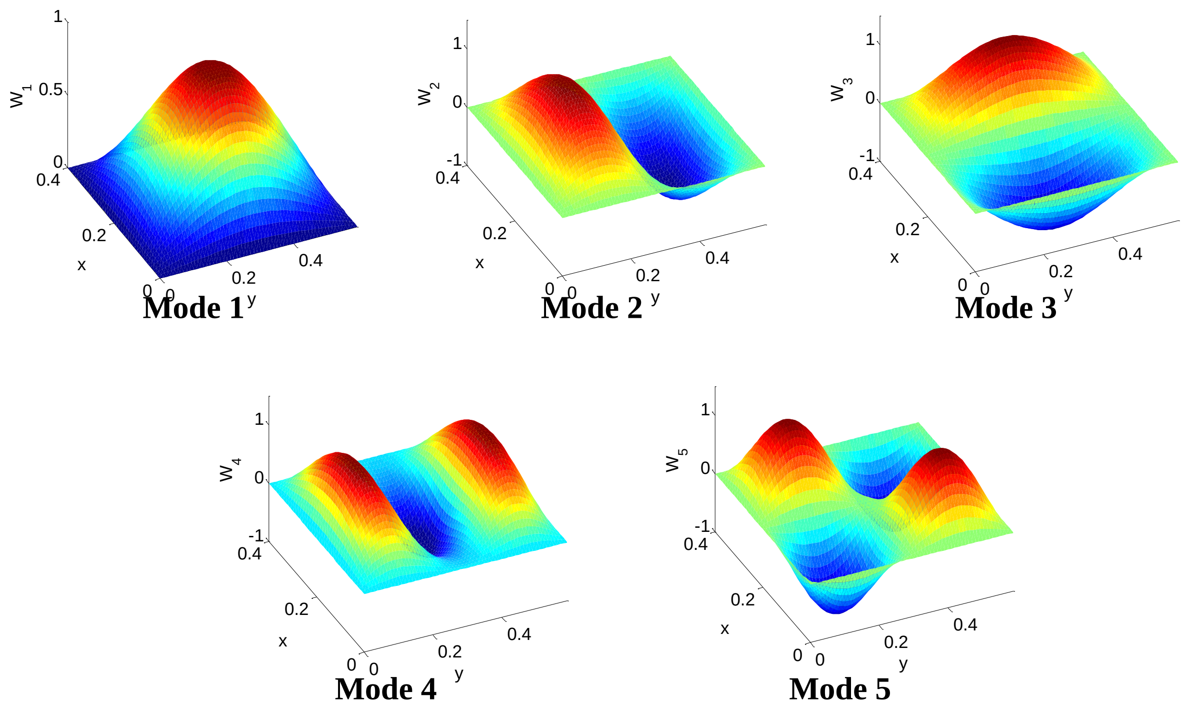

Figure 7. The first five modal shape of the rectangular plate carrying a distributed mass obtained from the FE model for the Case 1

V. CONCLUSIONS

In this paper, the free vibration of a rectangular plate carrying a uniformly distributed mass is investigated by experiments and finite element method, and the results are compared with those of the theoretical model. It is found that the theoretical frequencies are in good agreement with the experimental ones within an order of error 5-10 % and much closer to the FE results. For further comparison, the modal shapes of the structure obtained from the analytical model and the FE analysis are also given for one of the considered cases and they are also quite similar to each other. Thus, it is demonstrated that the theoretical approach in [5] give results which are highly accurate and consistent with the practice.

VI. FUTURE WORK

In a possible future work, it is aimed to study on the free vibration of the constrained rectangular plate on which two distributed mass are placed. This will allow controlling the eigenfrequencies and the modal shapes of the plate. Also, the theoretical model can be uploaded by considering the inertia effect of the attachment since this effect has an influence on the vibration response.

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