# BOUNDS FOR THE COMPLEX GROWTH RATE OF A PERTURBATION IN A COUPLE-STRESS FLUID IN THE PRESENCE OF MAGNETIC FIELD IN A POROUS MEDIUM

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## ABSTRACT

*A layer of couple-stress fluid heated from below in a porous medium is considered in the presence of uniform vertical magnetic field. Following the linearized stability theory and normal mode analysis, the paper through mathematical analysis of the governing equations of couple-stress fluid convection with a uniform vertical magnetic field in porous medium, for any combination of perfectly conducting free and rigid boundaries of infinite horizontal extension at the top and bottom of the fluid, established that the complex growth rate* $\sigma$ *of oscillatory perturbations, neutral or unstable for all wave numbers, must lie inside a semi-circle* $|\sigma|^2 = \left[\frac{R}{Ep_1}\left\{\frac{\varepsilon P_l p_2}{\varepsilon p_2(1+2\pi^2 F)+P_l\pi^2}\right\}\right]^2$ *in the right half of a complex* $\sigma$*-plane, Where R is the thermal Rayleigh number, F is the couple-stress parameter of the fluid,* $P_l$ *is the medium permeability,* $\varepsilon$ *is the porosity of the porous medium,* $p_1$ *is the thermal Prantl number and* $p_2$ *is the magnetic Prandtl number, which prescribes the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude in the couple-stress fluid heated from below in the presence of uniform vertical magnetic field in a porous medium. The result is important since the exact solutions of the problem investigated in closed form, are not obtainable for any arbitrary combinations of perfectly conducting dynamically free and rigid boundaries.*

***KEYWORDS:*** *Thermal convection; Couple-Stress Fluid; Magnetic field; PES; Chandrasekhar number.*

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## I. INTRODUCTION

Right from the conceptualizations of turbulence, instability of fluid flows is being regarded at its root. A detailed account of the theoretical and experimental study of the onset of thermal instability (Bénard Convection) in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [1] and the Boussinesq approximation has been used throughout, which states that the density changes are disregarded in all other terms in the equation of motion, except in the external force term. The formation and derivation of the basic equations of a layer of fluid heated from below in a porous medium, using the Boussinesq approximation, has been given in a treatise by Joseph [2]. When a fluid permeates through an isotropic and homogeneous porous medium, the gross effect is represented by Darcy's law. The study of layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can name the food processing industry, the chemical processing industry, solidification, and the centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in a porous medium. Stommel and Fedorov [3] and Linden [4] have remarked that the length scales characteristic of double-diffusive convecting layers in the ocean may be sufficiently large so that the Earth's rotation might be important in their formation. Moreover, the rotation of the Earth distorts the boundaries of a hexagonal convection cell in a fluid through porous medium, and this distortion plays an important role in the extraction of energy in geothermal regions. The forced convection in a fluid saturated porous medium channel has been studied by Nield et al [5]. An extensive and updated account of convection in porous media has been given by Nield and Bejan [6].

The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of the earth's core, where the earth's mantle, which consist of conducting fluid, behaves like a porous medium that can become conductively unstable as result of differential diffusion. Another application of the results of flow through a porous medium in the presence of magnetic field is in the study of the stability of convective geothermal flow. A good account of the effect of rotation and magnetic field on the layer of fluid heated from below has been given in a treatise by Chandrasekhar [1].

MHD finds vital applications in MHD generators, MHD flow-meters and pumps for pumping liquid metals in metallurgy, geophysics, MHD couplers and bearings, and physiological processes such magnetic therapy. With the growing importance of non-Newtonian fluids in modern technology and industries, investigations of such fluids are desirable. The presence of small amounts of additives in a lubricant can improve bearing performance by increasing the lubricant viscosity and thus producing an increase in the load capacity. These additives in a lubricant also reduce the coefficient of friction and increase the temperature range in which the bearing can operate.

Darcy's law governs the flow of a Newtonian fluid through an isotropic and homogeneous porous medium. However, to be mathematically compatible and physically consistent with the Navier-Stokes equations, Brinkman [7] heuristically proposed the introduction of the term $\frac{\mu}{\varepsilon}\nabla^2 \vec{q}$, (now known as Brinkman term) in addition to the Darcian term $-\left(\frac{\mu}{k_1}\right)\vec{q}$. But the main effect is through the Darcian term; Brinkman term contributes very little effect for flow through a porous medium. Therefore, Darcy's law is proposed heuristically to govern the flow of this non-Newtonian couple-stress fluid through porous medium. A number of theories of the micro continuum have been postulated and applied (Stokes [8]; Lai et al [9]; Walicka [10]). The theory due to Stokes [8] allows for polar effects such as the presence of couple stresses and body couples. Stokes's [8] theory has been applied to the study of some simple lubrication problems (see e.g. Sinha et al [11]; Bujurke and Jayaraman [12]; Lin [13]). According to the theory of Stokes [8], couple-stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [14] modeled synovial fluid as couple stress fluid in human joints. The study is motivated by a model of synovial fluid. The synovial fluid is natural lubricant of joints of the vertebrates. The detailed description of the joints lubrication has very important practical implications; practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The external efficiency of the physiological joint lubrication is caused by more mechanisms. The synovial fluid is caused by the content of the hyaluronic acid, a fluid of high viscosity, near to a gel. A layer of such fluid heated from below in a porous medium under the action of magnetic field and rotation may find applications in physiological processes. MHD finds applications in physiological processes such as magnetic therapy; rotation and heating may find applications in physiotherapy. The use of magnetic field is being made for the clinical purposes in detection and cure of certain diseases with the help of magnetic field devices.

Sharma and Thakur [15] have studied the thermal convection in couple-stress fluid in porous medium in hydromagnetics. Sharma and Sharma [16] have studied the couple-stress fluid heated from below in porous medium. Kumar and Kumar [17] have studied the combined effect of dust particles, magnetic field and rotation on couple-stress fluid heated from below and for the case of stationary convection, found that dust particles have destabilizing effect on the system, where as the rotation is found to have stabilizing effect on the system, however couple-stress and magnetic field are found to have both stabilizing and destabilizing effects under certain conditions. Sunil et al. [18] have studied the global stability for thermal convection in a couple-stress fluid heated from below and found couple-stress fluids are thermally more stable than the ordinary viscous fluids.

Pellow and Southwell [19] proved the validity of PES for the classical Rayleigh-Bénard convection problem. Banerjee et al [20] gave a new scheme for combining the governing equations of thermohaline convection, which is shown to lead to the bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries and, Banerjee and Banerjee [21] established a criterion on characterization of non-oscillatory motions in hydrodynamics which was further extended by Gupta et al. [22]. However no such result existed for non-Newtonian fluid configurations, in general and for couple-stress fluid configurations, in particular. Banyal [23] have characterized the non-oscillatory motions in couple-stress fluid. Banyal and Singh [24] has found the bounds for complex growth rate in the presence of uniform vertical rotation and Banyal and Khanna [25] in the presence of uniform vertical magnetic field.

Keeping in mind the importance of couple-stress fluids and magnetic field in porous media, as stated above,, the present paper is an attempt to prescribe the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude, in a layer of incompressible couple-stress fluid in a porous medium heated from below, in the presence of uniform vertical magnetic field, opposite to force field of gravity, when the bounding surfaces are of infinite horizontal extension, at the top and bottom of the fluid and are perfectly conducting with any combination of dynamically free and rigid boundaries.

The result is important since the exact solutions of the problem investigated in closed form, are not obtainable, for any arbitrary combination of perfectly conducting dynamically free and rigid boundaries.

This paper is organized as follows. In section 2, the linearized perturbation equations governing the present configuration are described. In section 3, using normal analysis the linearized perturbation equations are expressed in non-dimensional form, the there follows the mathematical analysis in section 4 and the bounds for the complex growth rate are derived. Finally we conclude our work in section 5.

## II. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we consider an infinite, horizontal, incompressible electrically conducting couple-stress fluid layer, of thickness d, heated from below so that, the temperature and density at the bottom surface z = 0 are $T_0$ and $\rho_0$ and at the upper surface z = d are $T_d$ and $\rho_d$ respectively, and that a uniform adverse temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ is maintained. The fluid is acted upon by a uniform vertical magnetic field $\vec{H}(0,0,H)$. This fluid layer is flowing through an isotropic and homogeneous porous medium of porosity $\varepsilon$ and of medium permeability $k_1$.

Let $\rho$, p, T, $\eta$, $\mu_e$ and $\vec{q}(u,v,w)$ denote respectively the fluid density, pressure, temperature, resistivity, magnetic permeability and filter velocity of the fluid, respectively Then the momentum balance, mass balance, and energy balance equation of couple-stress fluid and Maxwell's equations through porous medium, governing the flow of couple-stress fluid in the presence of uniform vertical magnetic field (Stokes [8]; Joseph [2]; Chandrasekhar [1]) are given by

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}\left(\vec{q}.\nabla\right)\vec{q}\right]=-\nabla\left(\frac{p}{\rho_o}\right)+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\vec{q}+\frac{\mu_e}{4\pi\rho_o}(\nabla\times\vec{H})\times\vec{H}, \quad (1)$$

$$\nabla.\vec{q}=0, \quad (2)$$

$$E\frac{dT}{dt}+(\vec{q}.\nabla)T=\kappa\nabla^2 T, \quad (3)$$

$$\nabla.\vec{H}=0, \quad (4)$$

$$\varepsilon\frac{d\vec{H}}{dt}=(\vec{H}.\nabla)\vec{q}+\varepsilon\eta\nabla^2\vec{H}, \quad (5)$$

Where $\frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon^{-1}\vec{q}.\nabla$, stands for the convective derivatives. Here

$E=\varepsilon+(1-\varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_v}\right)$, is a constant, while $\rho_s$, $c_s$ and $\rho_0$, $c_v$, stands for the density and heat capacity of the solid (porous matrix) material and the fluid, respectively, $\varepsilon$ is the medium porosity and $\vec{r}(x,y,z)$.

The equation of state is

$$\rho=\rho_0[1-\alpha(T-T_0)], \quad (6)$$

Where the suffix zero refer to the values at the reference level z = 0. Here $\vec{g}(0,0,-g)$ is acceleration due to gravity and $\alpha$ is the coefficient of thermal expansion. In writing the equation (1), we made use of the Boussinesq approximation, which states that the density variations are ignored in all terms in the equation of motion except the external force term. The kinematic viscosity $\nu$, couple-stress viscosity $\mu'$, magnetic permeability $\mu_e$, thermal diffusivity $\kappa$, and electrical resistivity $\eta$, and the coefficient of thermal expansion $\alpha$ are all assumed to be constants.

The basic motionless solution is

$$\vec{q}=(0,0,0) \quad , \quad \rho=\rho_0(1+\alpha\beta z), \quad p=p(z), \quad T=-\beta z+T_0, \quad (7)$$

Here we use the linearized stability theory and the normal mode analysis method. Assume small perturbations around the basic solution, and let $\delta\rho$, $\delta p$, $\theta$, $\vec{q}(u,v,w)$ and $\vec{h}=(h_x,h_y,h_z)$ denote respectively the perturbations in density $\rho$, pressure p, temperature T, velocity $\vec{q}(0,0,0)$ and the magnetic field $\vec{H}=(0,0,H)$. The change in density $\delta\rho$, caused mainly by the perturbation $\theta$ in temperature, is given by

$$\rho+\delta\rho=\rho_0[1-\alpha(T+\theta-T_0)]=\rho-\alpha\rho_0\theta, \text{ i.e. } \delta\rho=-\alpha\rho_0\theta. \quad (8)$$

Then the linearized perturbation equations of the couple-sress fluid reduces to

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t}=-\frac{1}{\rho_0}\nabla\delta p-\vec{g}\alpha\theta-\frac{1}{k_1}\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{h}\right)\times\vec{H}, \quad (9)$$

$$\nabla.\vec{q} = 0, \tag{10}$$

$$E\frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta, \tag{11}$$

$$\nabla.\vec{h} = 0, \tag{12}$$

$$\varepsilon \frac{\partial \vec{h}}{\partial t} = \left(\vec{H}.\nabla\right)\vec{q} + \varepsilon\eta\nabla^2 \vec{h}. \tag{13}$$

## III. NORMAL MODE ANALYSIS

Analyzing the disturbances into two-dimensional waves, and considering disturbances characterized by a particular wave number, we assume that the Perturbation quantities are of the form

$$[w, \theta, h_{z,}] = [W(z), \Theta(z), K(z)] \exp(ik_x x + ik_y y + nt), \tag{14}$$

Where $k_x, k_y$ are the wave numbers along the x- and y-directions, respectively, $k = (k_x^2 + k_y^2)^{\frac{1}{2}}$, is the resultant wave number, n is the growth rate which is, in general, a complex constant and, $W(z), \Theta(z)$ and $K(z)$ are the functions of z only.

Using (14), equations (9)-(13), Within the framework of Boussinesq approximations, in the non-dimensional form transform to

$$(D^2 - a^2)\left[\left(\frac{\sigma}{\varepsilon} + \frac{1}{P_l}\right) - \frac{F}{P_l}(D^2 - a^2)\right]W = -Ra^2\Theta + QD(D^2 - a^2)K, \tag{15}$$

$$(D^2 - a^2 - p_2\sigma)K = -DW, \tag{16}$$

and

$$(D^2 - a^2 - Ep_1\sigma)\Theta = -W, \tag{17}$$

Where we have introduced new coordinates $(x', y', z')$ = (x/d, y/d, z/d) in new units of length d and $D = d/dz'$. For convenience, the dashes are dropped hereafter. Also we have substituted $a = kd, \sigma = \frac{nd^2}{\nu}$, $p_1 = \frac{\nu}{\kappa}$, is the thermal Prandtl number; $p_2 = \frac{\nu}{\eta}$, is the magnetic Prandtl number; $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability, $F = \frac{\mu' /(\rho_0 d^2)}{\nu}$, is the dimensionless couple-stress viscosity parameter; $R = \frac{g\alpha\beta d^4}{\kappa\nu}$, is the thermal Rayleigh number and $Q = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon}$, is the Chandrasekhar number. Also we have Substituted $W = W_\oplus$, $\Theta = \left(\frac{\beta d^2}{\kappa}\right)\Theta_\oplus$, $K = \left(\frac{Hd}{\varepsilon\eta}\right)K_\oplus$, and $D_\oplus = dD$, and dropped $(\oplus)$ for convenience.

Now consider the case for any combination of the horizontal boundaries as, rigid-rigid or rigid-free or free-rigid or free-free at z=0 and z=1, as the case may be, and are perfectly conducting. The boundaries are maintained at constant temperature, thus the perturbations in the temperature are zero at the boundaries. The appropriate boundary conditions with respect to which equations (15)-(17), must possess a solution are

| | | |
|---|---|---|
| $W = 0 = \Theta$, | on both the horizontal boundaries, | (18) |
| DW=0, | on a rigid boundary, | (19) |

$D^2W = 0$, on a dynamically free boundary, (20)

K = 0, on both the boundaries as the regions outside the fluid are perfectly conducting, (21)

Equations (15)-(17) and appropriately adequate boundary conditions from (18)-(21), pose an eigenvalue problem for $\sigma$ and we wish to Characterize $\sigma_i$, when $\sigma_r \geq 0$.

## IV. MATHEMATICAL ANALYSIS

We prove the following theorems:

**Theorem 1:** If $R \rangle 0$, $F \rangle 0$, $Q \rangle 0$, $\sigma_r \geq 0$ and $\sigma_i \neq 0$ then the necessary condition for the existence of non-trivial solution $(W, \Theta, K)$ of equations (15) - (17) and the boundary conditions (18), (21) and any combination of (19) and (20) is that

$$|\sigma| \langle \left[ \frac{R}{Ep_1} \left\{ \frac{\varepsilon P_l p_2}{\varepsilon p_2 (1 + 2\pi^2 F) + P_l \pi^2} \right\} \right].$$

**Proof:** Multiplying equation (15) by $W^*$ (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$\left( \frac{\sigma}{\varepsilon} + \frac{1}{P_l} \right) \int_0^1 W^* (D^2 - a^2) W dz - \frac{F}{P_l} \int_0^1 W^* (D^2 - a^2)^2 W dz = -Ra^2 \int_0^1 W^* \Theta dz + Q \int_0^1 W^* D (D^2 - a^2) K dz, \quad (22)$$

Taking complex conjugate on both sides of equation (17), we get

$$(D^2 - a^2 - Ep_1 \sigma^*) \Theta^* = -W^*, \quad (23)$$

Therefore, using (23), we get

$$\int_0^1 W^* \Theta dz = -\int_0^1 \Theta (D^2 - a^2 - Ep_1 \sigma^*) \Theta^* dz, \quad (24)$$

Also taking complex conjugate on both sides of equation (16), we get

$$[D^2 - a^2 - p_2 \sigma^*] K^* = -DW^*, \quad (25)$$

Therefore, using (25) and using boundary condition (18), we get

$$\int_0^1 W^* D (D^2 - a^2) K dz = -\int_0^1 DW^* (D^2 - a^2) K dz = \int_0^1 K (D^2 - a^2)(D^2 - a^2 - p_2 \sigma^*) K^* dz, \quad (26)$$

Substituting (24) and (26) in the right hand side of equation (22), we get

$$\left( \frac{\sigma}{\varepsilon} + \frac{1}{P_l} \right) \int_0^1 W^* (D^2 - a^2) W dz - \frac{F}{P_l} \int_0^1 W^* (D^2 - a^2)^2 W dz$$

$$= Ra^2 \int_0^1 \Theta (D^2 - a^2 - Ep_1 \sigma^*) \Theta^* dz + Q \int_0^1 K (D^2 - a^2)(D^2 - a^2 - p_2 \sigma^*) K^* dz, \quad (27)$$

Integrating the terms on both sides of equation (27) for an appropriate number of times by making use of the appropriate boundary conditions (18) - (21), we get

$$\left( \frac{\sigma}{\varepsilon} + \frac{1}{P_l} \right) \int_0^1 \left\{ |DW|^2 + a^2 |W|^2 \right\} dz + \frac{F}{P_l} \int_0^1 \left( |D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2 \right) dz$$

$$= Ra^2 \int_0^1 \left\{ |D\Theta|^2 + a^2 |\Theta|^2 + Ep_1 \sigma^* |\Theta|^2 \right\} dz - Q \int_0^1 \left( |D^2 K|^2 + 2a^2 |DK|^2 + a^4 |K|^2 \right) dz - Qp_2 \sigma^* \int_0^1 \left( |DK|^2 + a^2 |K|^2 \right) dz. \quad (28)$$

And equating the real and imaginary parts on both sides of equation (28), and cancelling $\sigma_i (\neq 0)$ throughout from imaginary part, we get

$$\left( \frac{\sigma_r}{\varepsilon} + \frac{1}{P_l} \right) \int_0^1 \left\{ |DW|^2 + a^2 |W|^2 \right\} dz + \frac{F}{P_l} \int_0^1 \left( |D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2 \right) dz = Ra^2 \int_0^1 \left\{ |D\Theta|^2 + a^2 |\Theta| \right\} dz$$

$$-Q\int_0^1\left(\left|D^2K\right|^2+2a^2|DK|^2+a^4|K|^2\right)dz+\sigma_r\left[Ra^2Ep_1\int_0^1|\Theta|^2dz-Qp_2\int_0^1\left(|DK|^2+a^2|K|^2\right)dz\right]$$

(29)

and

$$\frac{1}{\varepsilon}\int_0^1\left\{|DW|^2+a^2|W|^2\right\}dz=-Ra^2Ep_1\int_0^1|\Theta|^2dz+Qp_2\int_0^1\left(|DK|^2+a^2|K|^2\right)dz, \quad (30)$$

Equation (30) implies that,

$$Ra^2Ep_1\int_0^1|\Theta|^2dz-Qp_2\int_0^1\left(|DK|^2+a^2|K|^2\right)dz, \quad (31)$$

is negative definite and also,

$$Q\int_0^1\left\{|DK|^2+a^2|K|^2\right\}dz\geq\frac{1}{\varepsilon p_2}\int_0^1|DW|^2dz, \quad (32)$$

We first note that since $W$, $\Theta$ and K satisfy $W(0)=0=W(1)$, $\Theta(0)=0=\Theta(1)$ and $K(0)=0=K(1)$ in addition to satisfying to governing equations and hence we have from the Rayleigh-Ritz inequality [24],

$$\int_0^1|DW|^2dz\geq\pi^2\int_0^1|W|^2dz \text{ and } \int_0^1|DK|^2dz\geq\pi^2\int_0^1|K|^2dz, \quad (33)$$

Further, multiplying equation (17) and its complex conjugate (23), and integrating by parts each term on right hand side of the resulting equation for an appropriate number of times and making use of boundary conditions on $\Theta$ namely $\Theta(0)=0=(1)$ along with (22), we get

$$\int_0^1\left|\left(D^2-a^2\right)\Theta\right|^2dz+2Ep_1\sigma_r\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz+E^2{p_1}^2|\sigma|^2\int_0^1|\Theta|^2dz=\int_0^1|W|^2dz,$$

(34)

since $\sigma_r\geq 0,\sigma_i\neq 0$ therefore the equation (34) gives,

$$\int_0^1\left|\left(D^2-a^2\right)\Theta\right|^2dz\langle\int_0^1|W|^2dz, \quad (35)$$

And

$$\int_0^1|\Theta|^2dz\langle\frac{1}{E^2{p_1}^2|\sigma|^2}\int_0^1|W|^2dz, \quad (36)$$

It is easily seen upon using the boundary conditions (18) that

$$\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz=\text{Real part of }\left\{-\int_0^1\Theta^*\left(D^2-a^2\right)\Theta dz\right\}\leq\left|\int_0^1\Theta^*\left(D^2-a^2\right)\Theta dz\right|,$$

$$\leq\int_0^1\left|\Theta^*\left(D^2-a^2\right)\Theta\right|dz\leq\int_0^1|\Theta^*|\left|\left(D^2-a^2\right)\Theta\right|dz,$$

$$=\int_0^1|\Theta|\left|\left(D^2-a^2\right)\Theta\right|dz\leq\left\{\int_0^1|\Theta|^2dz\right\}^{\frac{1}{2}}\left\{\int_0^1\left|\left(D^2-a^2\right)\Theta\right|^2dz\right\}^{\frac{1}{2}}, \quad (37)$$

(Utilizing Cauchy-Schwartz-inequality)

Upon utilizing the inequality (35) and (36), inequality (37) gives

$$\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz\leq\frac{1}{Ep_1|\sigma|}\int_0^1|W|^2dz, \quad (38)$$

Now R 〉 0, $P_l\rangle 0$, $\varepsilon\rangle 0, F\rangle 0$ and $\sigma_r\geq 0$, thus upon utilizing (31) and the inequalities (32),( (33) and (38), the equation (29) gives,

$$I_1 + a^2\left[\left(\frac{1}{P_l} + \frac{2\pi^2 F}{P_l} + \frac{\pi^2}{\varepsilon p_2}\right) - \frac{R}{Ep_1|\sigma|}\right]\int_0^1 |W|^2 dz \langle 0, \tag{39}$$

Where

$$I_1 = \left(\frac{\sigma_r}{\varepsilon} + \frac{1}{P_l}\right)\int_0^1 |DW|^2 dz + \frac{\sigma_r a^2}{\varepsilon}\int_0^1 |W|^2 dz + \frac{F}{P_l}\int_0^1 \left\{|D^2W|^2 + a^4|W|^2\right\}dz + Q\int_0^1 \left(|D^2K|^2 + a^2|DK|^2\right)dz,$$

is positive definite.

And therefore , we must have

$$|\sigma| \langle \left[\frac{R}{Ep_1}\left\{\frac{\varepsilon P_l p_2}{\varepsilon p_2(1+2\pi^2 F) + P_l \pi^2}\right\}\right], \tag{40}$$

Hence, if

$$\sigma_r \geq 0 \text{ and } \sigma_i \neq 0, \text{ then} |\sigma| \langle \left[\frac{R}{Ep_1}\left\{\frac{\varepsilon P_l p_2}{\varepsilon p_2(1+2\pi^2 F) + P_l \pi^2}\right\}\right].$$

And this completes the proof of the theorem.

## V. CONCLUSIONS

The inequality (40) for $\sigma_r \geq 0$ and $\sigma_i \neq 0$, can be written as

$$\sigma_r^2 + \sigma_i^2 \langle \left[\frac{R}{Ep_1}\left\{\frac{\varepsilon P_l p_2}{\varepsilon p_2(1+2\pi^2 F) + P_l \pi^2}\right\}\right]^2,$$

The essential content of the theorem, from the point of view of linear stability theory is that for the configuration of couple-stress fluid of infinite horizontal extension heated form below, having top and bottom bounding surfaces are of infinite horizontal extension, at the top and bottom of the fluid and are perfectly conducting with any arbitrary combination of dynamically free and rigid boundaries, in the presence of uniform vertical magnetic field parallel to the force field of gravity, the complex growth rate of an arbitrary oscillatory motions of growing amplitude, lies inside a semi-circle in the right half of the $\sigma_r\,\sigma_i$ - plane whose Centre is at the origin and radius is equal to $\left[\frac{R}{Ep_1}\left\{\frac{\varepsilon P_l p_2}{\varepsilon p_2(1+2\pi^2 F) + P_l \pi^2}\right\}\right]$, Where R is the thermal Rayleigh number, F is the couple-stress parameter of the fluid, $P_l$ is the medium permeability, $\varepsilon$ is the porosity of the porous medium, $p_1$ is the thermal Prandtl number and $p_2$ is the magnetic Prandtl number. The result is important since the exact solutions of the problem investigated in closed form, are not obtainable, for any arbitrary combinations of perfectly conducting dynamically free and rigid boundaries.

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