ROBUST KALMAN FILTERING FOR LINEAR DISCRETE TIME UNCERTAIN SYSTEMS

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ABSTRACT

In this paper, a robust finite-horizon Kalman filter is designed for discrete time-varying uncertain systems, basically the problem of finite horizon robust Kalman filtering for uncertain discrete-time systems is studied. The system under consideration is subject to time-varying norm-bounded parameter uncertainty in both the state and output matrices. The problem addressed is the design of linear (lters having an error variance with an optimized guaranteed upper bound [1] for any allowed uncertainty.

The Kalman Filter (KF) is a set of equations that provides an efficient recursive computational solution of the linear estimation problem.Basicaly it is a numerical method used to track a time-varying signal in the presence of noise. In this paper we study the robust Kalman filtering problem for linear uncertain discrete time-delay systems. Developing KF algorithms for such systems is an important problem to obtain optimal state estimates by utilizing the information on uncertainties and time-delays. First, designing of KF for nominal discrete-time systems is studied. Considering the covariance of the error in the estimation the KF algorithm is derived which is further tested on the system under consideration is subject to time-varying norm-bounded parameter uncertainty in both the state and output matrices[6].

KEYWORDS- *Uncertain discrete-time systems, Linear discrete time system, Robust Kalman filtering (RKF), Kalman filtering (KF), Kalman gain.*

I. INTRODUCTION

Filtering is desirable in many situations in engineering and embedded systems. A good filtering algorithm can remove the noise from electromagnetic signals while retaining the useful information. One of the fundamental problems in control and signal processing is the estimation of the state variables of a dynamic system through available noisy measurements. Basically the KF is an estimator which estimates the future state of a linear dynamic system from series of noisy measurement. It is the problem of estimating the instantaneous state of a linear system from a measurement of outputs that are linear combinations of the states but corrupted with Gaussian white noise [1]. It has come to be well recognized that the popular Kalman Filtering theory is very sensitive to system data and has poor performance robustness when a good system model is hard to obtain or the system drifts. Thus, the standard Kalman Filter may not be robust against modeling uncertainty and disturbances. This has motivated many studies on robust Kalman Filter design, which may probably yield a suboptimal solution with respect to the nominal system. Our interest in this paper is to tackle the filtering problem for a class of discrete time-delay uncertain systems. Necessary and sufficient conditions for the design of such a robust filter with an optimized upper bound for the error variance were given[3].

II. PROBLEM FORMULATION

In this paper firstly we use Kalman Filter to estimate the state of a linear discrete system defined by equation-(3.1) & (3.2) and it is observed that KF works well for linear systems without considering the time-delay[2].

One of the problems with the Kalman filter is that they may not robust against modeling uncertainties. The Kalman filter algorithm is the optimal filter for a system without uncertainties. The performance of a KF may be significantly degraded if the actual system model does not match the model on which the KF was based. Hence a new filter is proposed which addresses the uncertainties in process and measurement noise covariance defined by equations(4.1) & (4.2) and gives better results than the standard Kalman filter. So in coming sections it has been shown that the KF is the most widely used filter for its better performance. It is a very simple and intuitive concept with good computational efficiency. But when uncertainty and time delay is included then the performance of KF may degraded, so RKF is considered which is robust against large parameter uncertainty and time delay. In coming sections a performance comparison between RKF and KF is obtained for linear discrete-time uncertain system [5].

III. KALMAN FILTERING FOR LINEAR DISCRETE TIME SYSTEM

The Kalman filter (KF) is a tool that can estimate the variables of a wide range of processes. In mathematical terms we would say that a Kalman filter estimates the states of a linear system. The Kalman filter not only works well in practice, but it is theoretically attractive because it can be shown that of all possible filters, it is the one that minimizes the variance of the estimation error. The KF is an extremely effective and versatile procedure for combining noisy sensor outputs to estimate the state of the system with uncertain dynamics. When applied to a physical system, the observer or filter will be under the influence of two noise sources: (i) Process noise, (ii) Measurement noise [3].

In order to use a Kalman filter to remove noise from a signal, the process that we are measuring must be able to be described by a linear system.

We use a KF to estimate the state $x_k \in \Re^n$ of a discrete time controlled system. A linear system is described by the following two equations:

$$x_{k+1} = Ax_k + Bw_k \tag{3.1}$$

$$y_k = Cx_k + v_k \tag{3.2}$$

Where, $x_k \in \Re^n$ is the system state, $y_k \in \Re^m$ is the measured output, $w_k \in \Re^q$ is the process noise, $v_k \in \Re^p$ is the measurement noise[5].

The matrix $A \in \Re^{n\times n}$ is the dynamics matrix which relates the sate at time step k to the sate at time step $k+1$. The matrix $B \in \Re^{n\times 1}$ called noise matrix. The matrix $C \in \Re^{m\times m}$ relates the state measurement y_k.

3.1 The Kalman Filter Design

Suppose we have a linear discrete-time system given as equation (3.1) and (3.2), our objective is to design KF in the form of an equation (3.3), and determine a gain matrix which minimizes the mean square of the error e_k.

$$\hat{x}_{k+1} = A_f \hat{x}_k + k_f y_k \tag{3.3}$$

Note that the matrix A_f & K_f both are time varying matrices to be determined in order that the estimation error $e_k = x_k - \hat{x}_k$ is guaranteed to be smaller than a certain bound for all uncertainty matrices, i.e., the estimation error dynamics satisfies equation (2.4) [2]

$$\mathrm{E}\Big(\big(x_k - \hat{x}_k\big)\big(x_k - \hat{x}_k\big)^T\Big) \le S_k. \tag{3.4}$$

The matrix $S_k \in \Re^{l\times l}$ in the Equation (3.4) is the covariance of estimation error.

Now, we find a solution to the Kalman filtering problem over finite horizon $[0, N]$ will be given using a Discrete Riccati Equation (DRE) approach. For the purpose, consider the augmented vector [3],

$$\xi_k = \begin{bmatrix} x_k - \hat{x}_k \\ \hat{x}_k \end{bmatrix} = \begin{bmatrix} e_k \\ \hat{x}_k \end{bmatrix} \in k^{2n}, \tag{3.5}$$

$$\xi_{k+1} = \begin{bmatrix} x_{k+1} - \hat{x}_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} e_{k+1} \\ \hat{x}_{k+1} \end{bmatrix}. \tag{3.6}$$

Using Equation (3.5)-(3.6) we get,

$$e_{k+1} = Ax_k + Bw_k - \left[A_f \hat{x}_k + K_f C x_k + K_f v_k \right] \tag{3.7}$$

Where $w_k \in \Re^q$ is the process noise, $v_k \in \Re^p$ is the measurement noise.
Further on solving equation (3.7) we get-

$$e_{k+1} = (A - K_f C)x_k - (A - K_f C)\hat{x}_k + (A - K_f C)\hat{x}_k + Bw_k - A_f \hat{x}_k - K_f v_k \tag{3.8}$$

$$\hat{x}_{k+1} = \left(A_f + K_f C\right)\hat{x}_k + K_f C e_k + K_f v_k \tag{3.9}$$

Using equation (3.8) and (3.9), we get the state space equation for the estimation error e_k as follows[8],

$$\xi_{k+1} = A_{c1}\xi_k + G\eta_k, \tag{3.10}$$

$$\xi_k = \begin{bmatrix} e_k \\ \hat{x}_k \end{bmatrix}, \eta_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix}, A_{c1} = \begin{bmatrix} A - K_f C & A - A_f - K_f C \\ K_f C & A_f - K_f C \end{bmatrix}$$

Where, $G = \begin{bmatrix} B & -K_f \\ 0 & K_f \end{bmatrix}.$

The covariance matrix of ξ_k of the error system satisfying the bound [4],

$$\xi\left[\xi_k \quad \xi_k^T\right] \le \Sigma_k, \forall K \in [0 \quad N] \tag{3.11}$$

after some algebraic manipulation we get

$$A_{c1}\Sigma_k A_{c1}^T + G\overline{W}G^T \le \Sigma_{k+1} \tag{3.12}$$

where $\overline{W} = diag\{W, V\}$, equation (3.12) is called matrix inequality equation.

Note that the optimal solution of Equation (3.12) should be of the following partitioned from [2].

$$\Sigma_k = \begin{bmatrix} \mathrm{E}(e_k e_k^T) & \mathrm{E}(e_k \hat{x}_k^T) \\ \mathrm{E}(\hat{x}_k e_k^T) & \mathrm{E}(\hat{x}_k \hat{x}_k^T) \end{bmatrix} = \begin{bmatrix} \Sigma_{11,k} & \Sigma_{12,k} \\ \Sigma_{21,k} & \Sigma_{22,k} \end{bmatrix} = \begin{bmatrix} S_k & 0 \\ 0 & P_k - S_k \end{bmatrix} \tag{3.13}$$

Since our main aim is to reduce covariance of estimation error $\Sigma_{11,k}$, using equation (3.11) & (3.13) we get,

$$\mathrm{E}\left(e_k e_k^T\right) \le \Sigma_{11,k} \forall k \in \left[0, N\right] \tag{3.14}$$

The A_f and Kalman gain K_f can be denoted as -

$$A_f = A - k_f C \tag{3.15}$$

$$K_f = \left(A Q_k C^T\right) R^{-1} \tag{3.16}$$

Ultimately we have $$S_{k+1} = AQ_k A^T - \left(AQ_k C^T\right) R^{-1} \left(AQ_k C^T\right)^T + BWB^T \tag{3.17}$$

According to all these result we can say that, the filter (3.3) is a quadratic estimator with an upper bound of error covariance S_k .[4]

IV. Robust Kalman Filtering for Linear Discrete-Time Uncertain Systems

A Robust Kalman Filter (RKF) is a robust version of the KF but with necessary modification to account for the parameter uncertainty. The RKF usually yields a suboptimal solution with respect to the nominal system but it guarantees an upper bound to the filtering error covariance in spite of large parameter uncertainties. The standard KF approach does not address the issue of robustness against large parameter uncertainty in the linear process model. The performance of a KF may be significantly degraded if the system having uncertainty.

In this chapter, we consider the problem of Robust Kalman Filtering for discrete-time systems with norm-bounded parameter uncertainty in both the state and output matrices [2].

4.1 Facts

(4.1.1) For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimension and $\Sigma_3^T \Sigma_3 \le I$, it follows that,
$\Sigma_1 \Sigma_2 \Sigma_3 + \Sigma_2^T \Sigma_3^T \Sigma_1^T \le \alpha^{-1} \Sigma_1 \Sigma_1^T + \Sigma_2^T \Sigma_2$.[1]

(4.1.2) Let Σ_1, Σ_2 and Σ_3 and $0 < R = R^T$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^T(t) H(t) \le I$. Then for any $\mu > 0$ satisfying $\mu \Sigma_2^T \Sigma_2 < R$, the following matrix inequality holds,
$\left(\Sigma_3 + \Sigma_1 H(t) \Sigma_2\right) R^{1} \left(\Sigma_3^T + \Sigma_2^T H(t) \Sigma_1^T\right) \le \mu^{-1} \Sigma_1 \Sigma_1^T + \Sigma_3 \left(R - \mu \Sigma_2^T \Sigma_2\right) \Sigma_3^T$.[1]

4.2 The Robust Kalman Filter Design for discrete - time uncertain system

We use a RKF to estimate the state $x_k \in \Re^n$ of a discrete time uncertain controlled system. The system is described by a linear stochastic difference equation as[3],

$$x_{k+1} = \left(A + \Delta A_k\right) x_k + B w_k \tag{4.1}$$

$$y = \left(C + \Delta C_k\right) x_k + v_k \tag{4.2}$$

In the following v_k and w_k will be regarded as zero mean, uncorrelated white noise sequence with covariance R_k and Q_k .

$$v_k = N\left(0, R_k\right) \tag{4.3}$$

$$w_k = N\left(0, Q_k\right) \tag{4.4}$$

In order to obtain good results from RKF we are assumed the uncertainty matrix in the following structure [2].

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E \tag{4.5}$$

where, $F_K \in \Re^{I \times J}$ is an unknown real time varying matrix and H_1, H_2 and E are known real constant matrices of appropriate dimensions that specify how the elements of A and C are affected by uncertainty in F_k.

Our objective is to design KF in the form of an Equation (4.6), and determine a gain matrix which minimize the mean square of the error e_k [10]'

$$\hat{x}_{k+1} = A_f \hat{x}_k + K_f \left[y_k - \left(C + \Delta C_{ek} \right) \hat{x}_k \right] \tag{4.6}$$

$$\Delta A_{ek} = \varepsilon_k A S_k E^T \left(I - \varepsilon E S_k E^T \right) E \tag{4.7}$$

$$\Delta C_{ek} = \varepsilon_k C S_k E^T \left(I - \varepsilon E S_k E^T \right) E \tag{4.8}$$

$$K_f = \left(A Q_k C^T + \varepsilon_k H_1 H_2^T \right) \left(C Q_k C^T + R_{\varepsilon k} \right)^{-1} \tag{4.9}$$

the estimation error $e_k = x_k - \hat{x}_k$ dynamics must satisfies equation (4.10),

$$\mathrm{E}\left(\left(x_k - \hat{x}_k \right) \left(x_k - \hat{x}_k \right)^T \right) \le S_k \tag{4.10}$$

The matrix $S_k \in \Re^{l \times l}$ in the equation (4.10) is the covariance of estimation error. On further simplifying we get,

$$e_{k+1} = \left(A + \Delta A_k\right) x_k + B w_k - \left\{ A_f \hat{x}_k + K_f \left[\left(C + \Delta C_k \right) x_k + v_k \right] \right\} \tag{4.11}$$

$$\hat{x}_{k+1} = K_f C e_k + \left(A_k - K_f C_k \right) \hat{x}_k + K_f \Delta C_k e_k + K_f \Delta C_k \hat{x}_k + K_f v_k \tag{4.12}$$

using equation (4.11) & (4.12) we get the state space equation for the estimation error e_k as,

$$\xi_k = \begin{bmatrix} e_k \\ \hat{x}_k \end{bmatrix}, \eta_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix}, A_{c1} = \begin{bmatrix} A - K_f C & A - A_f - K_f C \\ K_f C & A_f - K_f C \end{bmatrix}$$

where,

$$H_{c1} = \begin{bmatrix} H_1 - K_f H_2 \\ K_f H_2 \end{bmatrix}, G = \begin{bmatrix} B & -K_f \\ 0 & K_f \end{bmatrix}, E_{c1} = \begin{bmatrix} E & E \end{bmatrix}$$

The covariance matrix of ξ_k of the error system satisfying the bound,

$$\xi\left[\xi_{k+1} \quad \xi_{k+1}^T \right] \le \Sigma_{k+1}$$

OR

$$\mathrm{E}\left[\left\{ \left(A_{c1} + H_{c1} F_k E_{c1} \right) \xi_k + G\eta \right\} \left\{ \left(A_{c1} + H_{c1} F_k E_{c1} \right) \xi_k + G\eta \right\}^T \right] \le \Sigma_{k+1} \tag{4.13}$$

Our main aim is to reduce covariance of estimation error i.e S_k

$$\mathrm{E}\left(e_k e_k^T \right) \le \Sigma_{11,k} \forall k \in \left[0, N \right] \tag{4.14}$$

Now we are going to select the optimal value of A_f and K_f to minimize $tr\left(\Sigma_{11,k}\right)$, $\forall k \in [0, N]$,

$$A_f = (A - K_f C)\left[\left(I + P_k E^T \tilde{M}_k^{-1} E - \Sigma_{11,k} E^T \tilde{M}_k^{-1} E\right) \times \left(I + P_k E^T \tilde{M}_k^{-1} E\right)^{-1}\right]^{-1} \quad (4.15)$$

$$K_f = \left(AQ_k C^T + \varepsilon_k H_1 H_2^T\right)\left(CQ_k C^T + R_{\varepsilon k}\right)^{-1} \quad (4.16)$$

Using above equations we get,

$$\Sigma_{11,k+1} = AQA^T + \varepsilon_k^{-1} H_1 H_1^T + BWB^T - \left(AQC^T + \varepsilon_k^{-1} H_1 H_2^T\right)$$
$$\times\left(R_{\varepsilon k} + CQC^T\right)^{-1}\left(AQC^T + \varepsilon_k^{-1} H_1 H_2^T\right)^T \quad (4.17)$$

$$S_{k+1} = AQA^T + \varepsilon_k^{-1} H_1 H_1^T + BWB^T - \left(AQC^T + \varepsilon_k^{-1} H_1 H_2^T\right)$$
$$\times\left(R_{\varepsilon k} + CQC^T\right)^{-1}\left(AQC^T + \varepsilon_k^{-1} H_1 H_2^T\right)^T \quad (4.18)$$

According to all these result we can say that, the filter (4.6) is a quadratic estimator with an upper bound of error covariance S_k .[5]

V. EXPERIMENTAL RESULTS & SIMULATIONS

5.1 Simulation Result of Kalman Filter (KF)

Consider the following uncertain discrete time system[3],

$$x_{k+1} = \left\{\begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}\right\} x_k + \begin{bmatrix} -6 \\ 1 \end{bmatrix} w_k$$

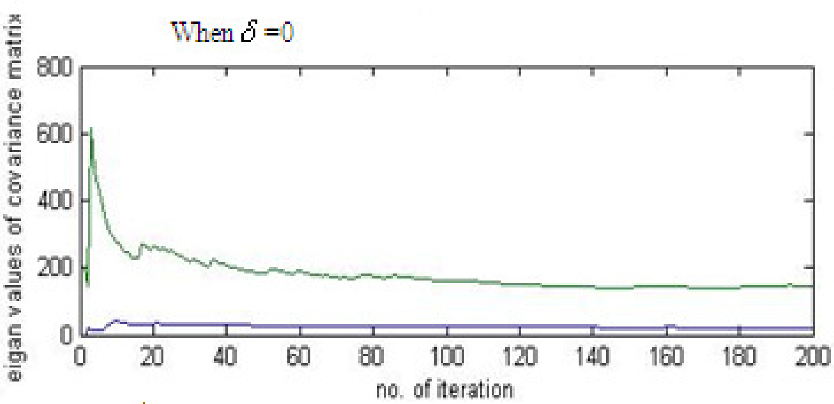
$$y_k = \begin{bmatrix} -100 & 10 \end{bmatrix} x_k + v_k$$

Where δ is an uncertain parameter satisfying $|\delta| \le 0.3$. Note that the above system is of the form of system (4.1)-(4.2) with[3],

$$H_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, H_2 = 0, E = \begin{bmatrix} 0 & 0.03 \end{bmatrix}$$

$$P = S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, W = V = 1$$

Now applying Kalman Filter for the above system ,the simulation results are as follows:



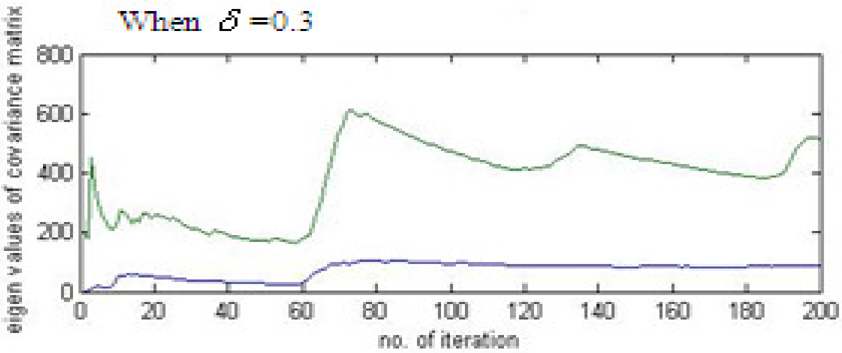


FIGURE 1: No. of Iteration vs. Eigen Values of Covariance Matrix with KF

5.2 Simulation Result of Robust Kalman Filter (RKF)

Applying Robust Kalman Filter for the same uncertain discrete time system as described in section 5.1, the simulation [12] results are as follows:

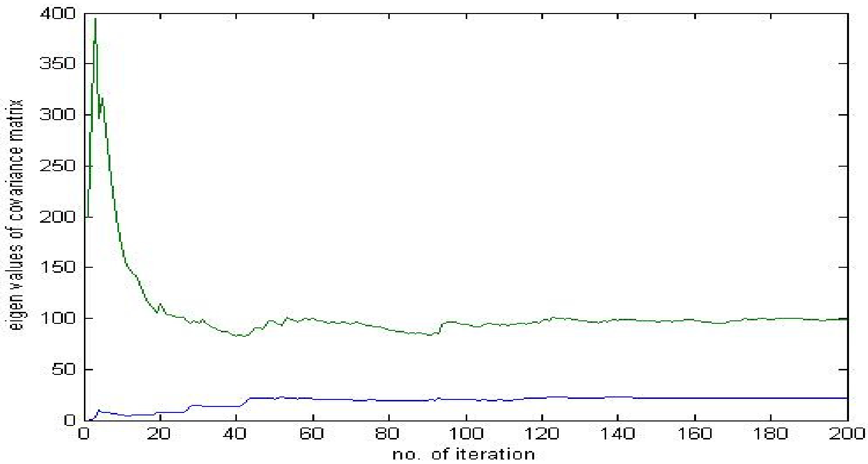


FIGURE 2: No. of Iteration vs. Eigen Values of Covariance Matrix with RKF

VI. CONCLUSION

In this paper, a formulation has been presented for the effective design of RKF for linear discrete-time uncertain system. Necessary and sufficient conditions for the design of robust quadratic filters are obtained. We have also analyzed that when the system has uncertainty, the RKF gives better performance as compare to KF. Results are illustrated by a numerical example.

Fig. 1 shows the graph between no. of iteration and Eigen values of covariance matrix, here when δ =0, the KF performance is better but, when δ=0.3, the KF performance is degraded. By increasing the uncertain parameter some spikes are included in system and it is difficult to KF to estimate the states.

Fig. 2 shows the performance of RKF it illustrate the graph between no. of iteration and Eigen values of covariance matrix, when δ=0.3. From the above said figures we can see that estimating the states with RKF is better than KF because the Eigen value of covariance matrix settles at 200 for δ=0 and gives poor performance in case of δ=0.3 but when RKF is considered for a system with δ=0.3 it gives better result than KF, as its eigen values settles at much lower level than of KF.

In this paper, we have given a solution to the problem of finite horizon robust Kalman filtering for uncertain discrete-time systems. We have also analyzed the feasibility and convergence properties of such robust filters. Results are illustrated by a numerical example.

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