NONLINEAR VIBRATIONS OF TIMOSHENKO BEAMS CARRYING A CONCENTRATED MASS

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ABSTRACT

Transverse vibrations of Timoshenko type beams carrying a concentrated mass have been investigated. Both ends of this mass-beam system have simply supports. Hamilton Principle has been used in order to derive equation of motion . For this coupled differential equations, approximately solutions have been searched by means of Method of Multiple Scales(a perturbation method). These solutions consist of two orders/parts; linear problem and nonlinear problem. One of them gives us natural frequency, and other one gives forced vibration solution. New symplectic method has been used to solve these coupled differential equations. Dynamic properties of the mass-beam system have been investigated using different control parameters; location and magnitude of the concentrated mass, rotational inertia and shear deformation effects.

KEYWORDS: *Timoshenko beam, symplectic approach, method of multiple scales, nonlinear vibrations.*

I. INTRODUCTION

A vast class of engineering problems which arise in industrial, civil, aero spatial, mechanical, electronic, medical, and automotive applications have been modeled as moving continua. Some models have been simplified into string, membrane or beam due to effects subjected to. While investigating transverse vibrations of these models, some assumptions have being done such as Euler-Bernoulli, Rayleigh, Timoshenko etc. beam theories. Studies using Euler-Bernoulli beam theory were reviewed by Nayfeh and Mook [45], and Nayfeh [46]. Some studies which carried out on Timoshenko beam theory [1], [2] are as follows: Taking into account various discontinuities which include cracks, boundaries and change in sections, Mei *et al.*[3] investigated axially loaded cracked Timoshenko beams. Deriving the transmission and reflection matrices of the beam, he examined relations between the injected waves and externally applied forces and moments. Loya *et al.*[4] handled problem of the cracked Timoshenko beams and obtained it natural frequencies. Using a new approach based on the dynamic stiffness solution, Banarjee [5] studied the free vibration problem of rotating Timoshenko beams. Using the Timoshenko and Euler–Bernoulli beam which replaced on elastic Winkler foundation, Ruge and Birk [6] examined dynamic stiffness coefficients related with the amplitude. Handling the Timoshenko beam, van Rensburg and van der Merwe [7] presented a systematic approach for Eigen-value problems associated with the system of partial differential equations. Hijmissen and van Horssen [8], investigated the transverse vibrations of the Timoshenko beam. They studied the influences of the beam parameters on decrease in magnitude of the frequencies. Majkut [9] derived a method a single differential equation of the fourth order which describes free and forced vibrations of a Timoshenko beam. Gunda [10] investigated effects of transverse shear and rotary inertia on vibration of the uniform Timoshenko beams. Shahba *et al.*[11] studied axially functionally graded tapered Timoshenko beams. Rossi *et al.*[12] examined analytical and exact solution of the Timoshenko beam model. For different supporting configuration, they found frequency coefficients. Geist and Mclaughlin [13], studied uniform Timoshenko beam with free ends. They gave a necessary and sufficient condition for determining eigenvalues for which there exist two linearly independent eigenfunctions. Esmailzadeh and Ohadi [14] studied non-uniform Timoshenko beam subjected to axial and tangential loads. They investigated frequency behavior of uniform and non-uniform beams with various boundary conditions; clamped supported, elastically supported, free end mass and pinned end mass. Zhong and Guo [15] investigated large-amplitude vibrations of simply supported Timoshenko beams with immovable ends. They studied on direct solution of the governing differential equations. Grant [16] examined uniform beams carrying a concentrated mass. For different end conditions, he investigated cross-sectional effects and effects of concentrated mass on frequency. Abramovich and Hamburger [17], studied a cantilever beam with a tip mass, examined the influence of rotary inertia and shear deformation on the natural frequencies of the system. Abramovich and Hamburger [18] studied uniform cantilever Timoshenko beam with a tip mass. They investigated the influence of rotary inertia and shear deformation on the natural frequencies of the beam. Chan and Wang [19] examined the problem of a Timoshenko beam partially loaded with distributed mass at an arbitrary position. They presented computational results on frequency variations. Cha and Pierre [20] studied Timoshenko beams with lumped attachments. They used a novel approach to determine the frequency equations of the combined dynamical system. Chang [21] studied simply supported beam carrying a rigid mass at the middle. Neglecting the effect of transverse shear deformation, he found general solution including both the rotatory inertia of the beam and of the concentrated mass. Lin [22] studied multi-span Timoshenko beam carrying multiple point masses, rotary inertias, linear springs, rotational springs and spring–mass systems. He investigated its free vibration characteristics. Posiadala [23] presented the solution of the free vibration problem of a Timoshenko beam with additional elements attached. He showed the influence of the various parameters on the frequencies of the combined system. Wu and Chen [24] studied Timoshenko beam carrying multiple spring-mass systems. They obtained natural frequencies for different supporting conditions; clamped-free, simple-simple, clamped-clamped and clamped-simple. Lin and Tsai [25], handled multi-span beam carrying multiple spring–mass systems. They studied the effects of attached spring–mass systems on the free vibration characteristics. Free vibration of a multi-span Timoshenko beam carrying multiple spring-mass systems has been studied by Yesilce *et al.*[26]. Later axial force effect in this multiple spring-mass systems has been investigated by Yesilce and Demirdag [27]. Using numerical assembly technique, Yesilce [41] studied vibrations of an axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements. Mei [42] studied the effects of lumped end mass on vibrations of a Timoshenko beam. The effects of lumped end mass on bending vibrations of Timoshenko beam has been investigated. Dos Santos and Reddy [43] studied free vibration analysis of Timoshenko beams and compared natural frequencies of the beam among classical elasticity, non-local elasticity, and modified couple stress theories. Stojanović and Kozić [44] studied vibration and buckling of a Rayleigh and Timoshenko double-beam system continuously joined by a Winkler elastic layer under compressive axial loading. They found general solutions of forced vibrations of beams subjected to arbitrarily distributed continuous loads. Li *et al.*[28] investigated nonlinear transverse vibrations of axially moving Timoshenko beams with two free ends. For the case of without internal resonances, they examined the relationships between the nonlinear frequencies and the initial amplitudes at different axial speeds and the nonlinear coefficients. Wu and Chen [29], investigated free and forced vibration responses for a uniform cantilever beam carrying a number of "spring damper-mass" systems. Maiz *et al.*[30] studied to determine the natural frequencies of vibration of a Bernoulli–Euler beam carrying a finite number of masses at arbitrary positions, having into account their rotatory inertia. Recently, Ghayesh *et al.*[31.32] developed a general solution procedure for nonlinear vibrations of beams with intermediate elements.

Background of the new symplectic method is as follows; Most recently, Lim *et al.*[33,34] proposed a new symplectic approach for the bending analysis of thin plates with two opposite edges simply supported. In their analysis, a series of bending moment functions were introduced to construct the Pro–Hellinger–Reissner variational principle, which is an analogy to plane elasticity. As for vibration analysis of plates, Zou [35] reported an exact symplectic geometry solution for the static and dynamic analysis of Reissner plates, but it was not exactly the same as the symplectic elasticity approach described above because trial mode shape functions for the simply supported opposite edges were still adopted in his analysis. To derive the exact free vibration solutions of moderately thick rectangular plates, Li and Zhong [36] proposed a new symplectic approach. Using new symplectic method and taking the type of the beam as Euler, Sarıgül and Boyacı [37] presented primary resonance of axially moving beams carrying a concentrated mass.

In this study, transverse vibrations of Timoshenko beam carrying a concentrated mass were handled. In section 2, problem being handled has been defined, parameters affecting on it determined and equations of motion has been obtained by using Hamilton Principle, which is well-known Energy Approach. In section 3, analytical solutions have been searched by means of Method of Multiple Scales (a perturbation method) under assumption of primary resonance. New Symplectic Method has been proposed to solve coupled differential equations. In section 4, numerical results has been obtained for different mass ratios, mass locations, shear correction coefficients, and rotational inertia effects. From natural frequencies and frequency–amplitude curves, vibrational characteristics of the Timoshenko type beam carrying a concentrated mass. For compatibility, some comparisons have been done with studies from Özkaya *et al.*[38-39] and Pakdemirli *et al.*[40].

II. PROBLEM FORMULATION

Transversally vibrating beam using Timoshenko theory has been drawn in Fig.1. The study could be seen a beam-mass system with simply supports. *M* concentrated mass is placed on the beam arbitrarily along *L* distance. The model with 1 mass is made of 2 parts. In order to formulate the model mathematically, energy of the system has been used by means of Hamilton's principle. The whole system consists of kinetic (*T*) and potential (*U*) as shown below;

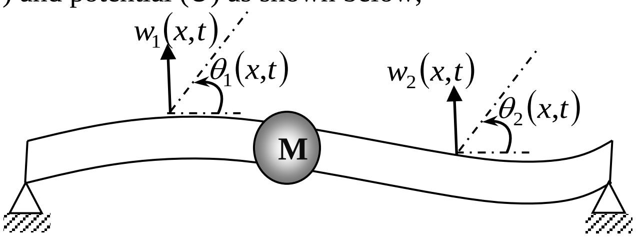

Figure 1. Timoshenko beam carrying a concentrated mass.

$$\delta\int_{t_1}^{t_2}(T-U)\,dt=0\,, \tag{1}$$

$$U=\int_0^{x_s}\frac{1}{2}E\,A\,\varepsilon_1^2\,dx+\int_{x_s}^{L}\frac{1}{2}E\,A\,\varepsilon_2^2\,dx+\int_0^{x_s}\frac{1}{2}E\,I\,\kappa_1^2\,dx+\int_{x_s}^{L}\frac{1}{2}E\,I\,\kappa_2^2\,dx+\int_0^{x_s}\frac{1}{2}k\,G\,A\,\gamma_1^2\,dx+\int_{x_s}^{L}\frac{1}{2}k\,G\,A\,\gamma_2^2\,dx \tag{2}$$

$$T=\int_0^{x_s}\frac{1}{2}\rho\,A\left(\frac{\partial w_1}{\partial t}\right)^2dx+\int_{x_s}^{L}\frac{1}{2}\rho\,A\left(\frac{\partial w_2}{\partial t}\right)^2dx+\int_0^{x_s}\frac{1}{2}\rho\,J\left(\frac{\partial\theta_1}{\partial t}\right)^2dx+\int_{x_s}^{L}\frac{1}{2}\rho\,J\left(\frac{\partial\theta_2}{\partial t}\right)^2dx+\frac{1}{2}M\left(\frac{\partial w_1(x_s,t)}{\partial t}\right)^2 \tag{3}$$

where $\partial/\partial t$ and $\partial/\partial x$ denote partial differentiations with respect to the time *t*, and the spatial variable *x*, respectively. *w* is the transverse displacement and θ is its slope, ρ is the mass density per unit volume, *A* is the cross section area of the beam, *I* and *J* are is the moments of the inertia, *E* is the Young's modulus, *G* is the shear modulus, and *k* is shear correction coefficient, respectively.
It is assumed that Timoshenko beams deform within linear elastic regime and therefore Hooke's law is valid. The nonlinear membrane strain-displacement, bending curvature-displacement and shear strain-displacement relations of the beam are given as;

$$\varepsilon_i=\frac{\partial u_i}{\partial x}+\frac{1}{2}\left(\frac{\partial w_i}{\partial x}\right)^2,\quad \kappa_i=\frac{\partial\theta_i}{\partial x}\Big/\sqrt{1+\left(\frac{\partial w_i}{\partial x}\right)^2}\approx\frac{\partial\theta_i}{\partial x}-\frac{1}{2}\frac{\partial\theta_i}{\partial x}\left(\frac{\partial w_i}{\partial x}\right)^2,\quad \gamma_i=\tan^{-1}\left(\frac{\partial w_i}{\partial x}\right)-\theta_i\approx\frac{\partial w_i}{\partial x}-\theta_i\,, \tag{4}$$

where *u, w,* θ*,* ε*,* κ and γ represent the axial displacement, the deflection, the cross-section rotation, the membrane strain, the bending curvature, and the shear strain, respectively.

Before processing, we must present following dimensionless quantities under notation *i*=1,2;

$$\hat{w}_i(\hat{x},\hat{t})=\frac{w_i(x,t)}{r},\quad \hat{u}_i(\hat{x},\hat{t})=\frac{u_i(x,t)}{L},\quad \hat{\theta}_i(\hat{x},\hat{t})=\frac{L}{r}\theta_i(x,t),\quad I=Ar^2,\quad J=AL^2,\quad \hat{x}=\frac{x}{L},\ \eta=\frac{x_s}{L},\ \hat{t}=t\Big/\sqrt{\frac{\rho AL^4}{EI}} \tag{5}$$

where η is the dimensionless mass location, and *r* is the radius of gyration of the beam cross section.
By means of Hamilton's principle, one can substitute Eqs.(2-3) into Eq.(1) and performing necessary calculations it is seen that longitudinal terms($\hat{u}_i$) can be eliminated from the equations. Adding dimensionless damping ($\hat{\mu}_i$) and forcing terms ($\hat{F}$) into remained equations in process, one can obtain the dimensionless form of the equations of motion;

$$\chi\,\nu\,\frac{\partial}{\partial\hat{x}}\left(\frac{\partial\hat{w}_i}{\partial\hat{x}}-\theta_i\right)+\frac{1}{2}\left\{\int_0^{\eta}\left(\frac{\partial\hat{w}_1}{\partial\hat{x}}\right)^2 d\hat{x}+\int_{\eta}^{1}\left(\frac{\partial\hat{w}_2}{\partial\hat{x}}\right)^2 d\hat{x}\right\}\frac{\partial^2\hat{w}_i}{\partial\hat{x}^2}-\frac{1}{\nu}\frac{\partial}{\partial\hat{x}}\left\{\left(1-\frac{1}{2}\frac{1}{\nu}\left(\frac{\partial\hat{w}_i}{\partial\hat{x}}\right)^2\right)\left(\frac{\partial\hat{\theta}_i}{\partial\hat{x}}\right)^2\frac{\partial\hat{w}_i}{\partial\hat{x}}\right\}+\hat{F}_i\cos(\Omega.t)=\frac{\partial^2\hat{w}_i}{\partial\hat{t}^2}+\hat{\mu}_i\frac{\partial\hat{w}_i}{\partial\hat{t}}$$

$$\chi\,\nu\left(\frac{\partial\hat{w}_i}{\partial\hat{x}}-\theta_i\right)+\left(1-\frac{1}{2}\frac{1}{\nu}\left(\frac{\partial\hat{w}_i}{\partial\hat{x}}\right)^2\right)^2\frac{\partial^2\hat{\theta}_i}{\partial\hat{x}^2}+\frac{\partial}{\partial\hat{x}}\left\{\left(1-\frac{1}{2}\frac{1}{\nu}\left(\frac{\partial\hat{w}_i}{\partial\hat{x}}\right)^2\right)^2\right\}\frac{\partial\hat{\theta}_i}{\partial\hat{x}}=\frac{\partial^2\hat{\theta}_i}{\partial\hat{t}^2},\quad i=1,2. \tag{6}$$

Matching and boundary conditions could be written as follows;

$$\hat{w}_1=\frac{\partial\hat{\theta}_1}{\partial\hat{x}}=0\ \text{ for }\ \hat{x}=0,\quad \hat{w}_2=\frac{\partial\hat{\theta}_2}{\partial\hat{x}}=0\ \text{ for }\ \hat{x}=1,$$

$$\chi.\nu.\left(\frac{\partial\hat{w}_1}{\partial\hat{x}}-\theta_1\right)+\frac{1}{2}.\left\{\int_0^{\eta}\left(\frac{\partial\hat{w}_1}{\partial\hat{x}}\right)^2 d\hat{x}+\int_{\eta}^{1}\left(\frac{\partial\hat{w}_2}{\partial\hat{x}}\right)^2 d\hat{x}\right\}.\frac{\partial\hat{w}_1}{\partial\hat{x}}-\frac{1}{\nu}\left(1-\frac{1}{2}\frac{1}{\nu}\left(\frac{\partial\hat{w}_1}{\partial\hat{x}}\right)^2\right).\left(\frac{\partial\hat{\theta}_1}{\partial\hat{x}}\right)^2.\frac{\partial\hat{w}_1}{\partial\hat{x}}+\alpha.\frac{\partial^2\hat{w}_1}{\partial\hat{t}^2}=$$

$$\chi.\nu.\left(\frac{\partial\hat{w}_2}{\partial\hat{x}}-\theta_2\right)+\frac{1}{2}.\left\{\int_0^{\eta}\left(\frac{\partial\hat{w}_1}{\partial\hat{x}}\right)^2 d\hat{x}+\int_{\eta}^{1}\left(\frac{\partial\hat{w}_2}{\partial\hat{x}}\right)^2 d\hat{x}\right\}.\frac{\partial\hat{w}_2}{\partial\hat{x}}-\frac{1}{\nu}.\left(1-\frac{1}{2}\frac{1}{\nu}\left(\frac{\partial\hat{w}_2}{\partial\hat{x}}\right)^2\right).\left(\frac{\partial\hat{\theta}_2}{\partial\hat{x}}\right)^2.\frac{\partial\hat{w}_2}{\partial\hat{x}},$$

$$\hat{w}_1=\hat{w}_2,\quad \hat{\theta}_1=\hat{\theta}_2,\quad \frac{\partial\hat{\theta}_1}{\partial\hat{x}}=\frac{\partial\hat{\theta}_2}{\partial\hat{x}}\quad \text{for }\ \hat{x}=\eta \tag{7}$$

Here, some simplifications have been done as follows

$$\alpha=\frac{M}{\rho\,A\,L},\qquad \nu=\frac{L}{r},\quad \chi=\frac{k\,G}{E},\qquad \frac{E}{G}=\frac{1}{2(1+\upsilon)}, \tag{8}$$

where α is the dimensionless mass parameter, ν is the slenderness ratio, υ is the poisson`s ratio and χ is the shear/flexural rigidity ratio.

III. Analytical Solutions

We apply the Method of Multiple Scales (MMS), a perturbation technique (see ref. [46]), directly to the partial-differential equations and its boundary and continuity conditions. After removing symbol of ^ for easy readability of equations and impending i=1,2 in this method, we assume expansions as follows;

$$\hat{w}_i(\hat{x},\hat{t})=\varepsilon\, w_{i1}(x,T_0,T_1,T_2)+\varepsilon^2\, w_{i2}(x,T_0,T_1,T_2)+\varepsilon^3\, w_{i3}(x,T_0,T_1,T_2),$$

$$\hat{\theta}_i(\hat{x},\hat{t})=\varepsilon\,\theta_{i1}(x,T_0,T_1,T_2)+\varepsilon^2\,\theta_{i2}(x,T_0,T_1,T_2)+\varepsilon^3\,\theta_{i3}(x,T_0,T_1,T_2) \tag{9}$$

where ε is a small book-keeping parameter artificially inserted into the equations. This parameter can be taken as 1 at the end upon keeping in mind, however, that deflections are small. Therefore, we investigated a weak nonlinear system. $T_0=t$ is the fast time scale, $T_1=\varepsilon\, t$ and, $T_2=\varepsilon^2\, t$ are the slow time scales in MMS.

Now consider only the primary resonance case and hence, the forcing and damping terms are ordered as $\hat{F}_i=\varepsilon^3 F_i$, $\hat{\mu}_i=\varepsilon^2\mu_i$ so that they counter the effect of the nonlinear terms. Derivatives with respect to time were written in terms of the T_n as follow:

$$\frac{\partial}{\partial t}=D_0+\varepsilon\, D_1+\varepsilon^2\, D_2,\ \frac{\partial^2}{\partial t^2}=D_0^2+2\,\varepsilon\,D_0\,D_1+\varepsilon^2\left(D_1^2+2\,D_0\,D_2\right),\ D_n\equiv\partial/\partial T_n. \tag{10}$$

3.1. Linear Problem

First order of Perturbation Method could be defined as linear problem. Substituting Eqs.(9)-(10) into Eqs.(6)-(7) and separating each order of ε, one obtains the followings;

order ε: $\chi\,\nu\left(w_{i1}''-\theta_{i1}'\right)-D_0^2 w_{i1}=0,\quad \chi\,\nu\left(w_{i1}'-\theta_{i1}\right)+\theta_{i1}''-D_0^2\theta_{i1}=0$

$$w_{11}|_{x=0}=\theta_{11}'|_{x=0}=0,\ w_{21}|_{x=1}=\theta_{21}'|_{x=1}=0,\ w_{11}|_{x=\eta}=w_{21}|_{x=\eta},\ \theta_{11}|_{x=\eta}=\theta_{21}|_{x=\eta},\ \theta_{11}'|_{x=\eta}=\theta_{21}'|_{x=\eta},$$

$$\chi\,\nu\left(w_{11}'-\theta_{11}\right)\Big|_{x=\eta}-\chi\,\nu\left(w_{21}'-\theta_{21}\right)\Big|_{x=\eta}+\alpha\,D_0^2 w_{11}\Big|_{x=\eta}=0 \tag{11}$$

order ε^2: $\chi\,\nu\left(w_{i2}''-\theta_{i2}'\right)-D_0^2 w_{i2}=2\,D_0 D_1\, w_{i1},\quad \theta_{i2}''+\chi\,\nu\left(w_{i2}'-\theta_{i2}\right)-D_0^2\theta_{i2}=2\,D_0 D_1\,\theta_{i1}$

$$w_{12}|_{x=0}=\theta_{12}'|_{x=0}=0,\quad w_{22}|_{x=1}=\theta_{22}'|_{x=1}=0,\ w_{12}|_{x=\eta}=w_{22}|_{x=\eta},\ \theta_{12}|_{x=\eta}=\theta_{22}|_{x=\eta},\ \theta_{12}'|_{x=\eta}=\theta_{22}'|_{x=\eta}$$

$$\chi\,\nu\left(w_{12}'-\theta_{12}\right)\Big|_{x=\eta}-\chi\,\nu\left(w_{22}'-\theta_{22}\right)\Big|_{x=\eta}+\alpha\,D_0^2 w_{12}\Big|_{x=\eta}=-2\,\alpha\,D_0 D_1\, w_{11}\Big|_{x=\eta} \tag{12}$$

order ε^3:

$$\chi\,\nu\left(w_{i3}''-\theta_{i3}'\right)-D_0^2 w_{i3}=2\,D_0\,D_1\,w_{i2}+\left(D_1^2+2\,D_0\,D_2\right)w_{i1}+\mu_i D_0 w_{i1}-\frac{1}{2}\left\{\int_0^\eta w_{11}'^2\,dx+\int_\eta^1 w_{21}'^2\,dx\right\}.w_{i1}''-F_i\cos(\Omega\,t)+\frac{1}{\nu}\left\{w_{i1}'\;\theta_{i1}'^2\right\}'$$

$$\theta_{i3}''+\chi\,\nu\left(w_{i3}'-\theta_{i3}\right)-D_0^2\theta_{i3}=2D_0\,D_1\,\theta_{i2}+\left(D_1^2+2\,D_0\,D_2\right)\theta_{i1}+\frac{1}{\nu}\left(w_{i1}'^2\theta_{i1}'\right)'$$

$$w_{13}\big|_{x=0}=\theta_{13}'\big|_{x=0}=0\,,\quad w_{23}\big|_{x=1}=\theta_{23}'\big|_{x=1}=0\,,\quad w_{13}\big|_{x=\eta}=w_{23}\big|_{x=\eta}\,,\quad \theta_{13}\big|_{x=\eta}=\theta_{23}\big|_{x=\eta}\,,\quad \theta_{13}'\big|_{x=\eta}=\theta_{23}'\big|_{x=\eta}\,,$$

$$\chi\,\nu\left(w_{13}'-\theta_{13}\right)\Big|_{x=\eta}-\chi\,\nu\left(w_{23}'-\theta_{23}\right)\Big|_{x=\eta}+\alpha\,D_0^2\,w_{13}\Big|_{x=\eta}=-2\,\alpha\,D_0\,D_1\,w_{12}\Big|_{x=\eta}-\alpha\left(D_1^2+2\,D_0 D_2\right)w_{11}\Big|_{x=\eta}$$

$$-\frac{1}{2}.\left\{\int_0^\eta w_{11}'^2\,dx+\int_\eta^1 w_{21}'^2\,dx\right\}\left(w_{11}'-w_{21}'\right)\Big|_{x=\eta}-\frac{1}{\nu}\left\{-\theta_{11}'^2\,w_{11}'+\theta_{21}'^2 w_{21}'\right\}\Big|_{x=\eta} \tag{13}$$

Linear problem is governed by Eq.(11) at order ε^1. For solution to the problem, following forms are assumed

$$w_{i1}(x,T_0,T_1,T_2)=A(T_1,T_2)\,e^{i\,\omega_1 T_0}\,Y_i(x)+\bar{A}(T_1,T_2)\,e^{-i\,\omega_1 T_0}\,\bar{Y}_i(x)\,,\;\; \theta_{i1}(x,T_0,T_1,T_2)=B(T_1,T_2)\,e^{i\,\omega_2 T_0}\,\phi_i(x)+\bar{B}(T_1,T_2)\,e^{-i\,\omega_2 T_0}\,\bar{\phi}_i(x) \tag{14}$$

where overbar denotes the complex conjugate of the expression. ω_1, Y, A represent natural frequency, eigenfunction and amplitude of the transverse term, respectively. And similarly ω_2, ϕ, B represent frequency, eigenfunction and amplitude of the rotational term, respectively.

Substituting Eq.(14) into Eq.(11), one obtains following equations which satisfies the mode shapes:

$$\chi\,\nu\left\{Ae^{i\,\omega_1 T_0}\,Y_i''-B\,e^{i\,\omega_2 T_0}\,\phi_i'\right\}+\omega_1^2\,A\,e^{i\,\omega_1 T_0}\,Y_i=0\,,\quad \chi\,\nu\left\{Ae^{i\,\omega_1 T_0}\,Y_i'-B\,e^{i\,\omega_2 T_0}\,\phi_i\right\}+\omega_2^2\,B\,e^{i\,\omega_2 T_0}\,\phi_i+B\,e^{i\,\omega_2 T_0}\,\phi_i''=0$$

$$Y_1\big|_{x=0}=\phi_1'\big|_{x=0}=0\,,\quad Y_2\big|_{x=1}=\phi_2'\big|_{x=1}=0\,,\quad Y_1\big|_{x=\eta}=Y_2\big|_{x=\eta}\,,\quad \phi_1\big|_{x=\eta}=\phi_2\big|_{x=\eta}\,,\quad \phi_1'\big|_{x=\eta}=\phi_2'\big|_{x=\eta}$$

$$\chi\,\nu\left(A\,e^{i\,\omega_1 T_0}\,Y_1'-B\,e^{i\,\omega_2 T_0}\,\phi_1\right)\Big|_{x=\eta}-\chi\,\nu\left(A\,e^{i\,\omega_1 T_0}\,Y_2'-B\,e^{i\,\omega_2 T_0}\,\phi_2\right)\Big|_{x=\eta}-\alpha\,\omega_1^2\,A\,e^{i\,\omega_1 T_0}\,Y_1\big|_{x=\eta}=0 \tag{15}$$

Complex conjugates of the mode shapes are the same for both transverse and rotational terms. Thus, there is no need to write the complex conjugate equations (cc).

3.2. Non-Linear Problem

Adding the additive of the other orders according to the first order gives us non-lınear problem. In order to propose a solution at order ε^2, $D_1 w_{i1}=0$ and $D_1\theta_{i1}=0$ must be done. Thus, the form of differential equations and its boundary and continuous conditions at order ε^2 are same of order ε^1. Also, this means $A=A(T_2)$, $B=B(T_2)$. Thanks to perturbation method, order ε^2 was neglected and according to Eq.(12), following equations at order ε^3 were obtained

$$\chi\,\nu\left(w_{i3}''-\theta_{i3}'\right)-D_0^2\,w_{i3}=-F_i\cos(\Omega\,t)$$

$$+e^{i\omega_1 T_0}\left\{2\,i\,\omega_1\,\dot{A}\,Y_i+2\,i\,\omega_1\,\mu_i\,A\,Y_i+2\,\frac{1}{\nu}\,\bar{B}\,B\,A\left(\bar{\phi}_i'\,\phi_i'\,Y_i\right)'+A^2\,\bar{A}\left[-\left\{\int_0^\eta \bar{Y}_1'Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}Y_i''-\frac{1}{2}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\bar{Y}_i''\right]\right\}$$

$$+e^{-i\omega_1 T_0}\left[-2\,i\,\omega_1\,\dot{\bar{A}}\,\bar{Y}_i-2\,i\,\omega_1\,\mu_i\,\bar{A}\,\bar{Y}_i+2\,\frac{1}{\nu}\,\bar{B}\,B.\bar{A}\left(\bar{\phi}_i'\,\phi_i'\,\bar{Y}_i\right)'+\bar{A}^2\,A\left[-\frac{1}{2}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}Y_i''-\left\{\int_0^\eta \bar{Y}_1'\,Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}\bar{Y}_i''\right]\right]$$

$$+\frac{1}{\nu}\,B^2\,A\,e^{i(\omega_1+2\omega_2)T_0}\left(\phi_i'^2\,Y_i\right)'+\frac{1}{\nu}\,\bar{B}^2\,\bar{A}\,e^{-i(\omega_1+2\omega_2)T_0}\left(\bar{\phi}_i'^2\,\bar{Y}_i\right)'+\frac{1}{\nu}\,\bar{B}^2 A\,e^{i(\omega_1-2\omega_2)T_0}\left(\bar{\phi}_i'^2\,Y_i\right)'+\frac{1}{\nu}\,B^2\,\bar{A}\,e^{-i(\omega_1-2\omega_2)T_0}\left(\phi_i'^2\,\bar{Y}_i\right)'$$

$$-\frac{1}{2}\,A^2\,A\,e^{3i\,\omega_1 T_0}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}Y_i''-\frac{1}{2}\,\bar{A}^2\,\bar{A}\,e^{-3i\,\omega_1 T_0}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\bar{Y}_i''$$

$$\theta_{i3}''+\chi\,\nu\left(w_{i3}'-\theta_{i3}\right)-D_0^2\theta_{i3}=+e^{i\omega_2 T_0}\left\{2\,i\,\omega_2\,\dot{B}\,\phi_i+2\,\frac{1}{\nu}\,\bar{A}\,A\,B\left(\bar{Y}_i'Y_i'\,\phi_i'\right)'\right\}+e^{-i\omega_2 T_0}\left\{-2\,i\,\omega_2\,\dot{\bar{B}}\,\bar{\phi}_i+2\,\frac{1}{\nu}\,\bar{A}\,A\,\bar{B}.\left(\bar{Y}_i'Y_i'\,\bar{\phi}_i'\right)'\right\}$$

$$+\frac{1}{\nu}\,A^2\,B\,e^{i(2\omega_1+\omega_2)T_0}\left(Y_i'^2\,\phi_i'\right)'+\frac{1}{\nu}\,\bar{A}^2\,\bar{B}\,e^{-i(2\omega_1+\omega_2)T_0}\left(\bar{Y}_i'^2\,\bar{\phi}_i'\right)'+\frac{1}{\nu}\,\bar{A}^2\,B\,e^{-i(2\omega_1-\omega_2)T_0}\left(\bar{Y}_i'^2\,\phi_i'\right)'+\frac{1}{\nu}\,A^2\,\bar{B}\,e^{i(2\omega_1-\omega_2)T_0}\left(Y_i'^2\bar{\phi}_i'\right)'$$

$$w_{13}\big|_{x=0}=\theta_{13}'\big|_{x=0}=0\,,\; w_{23}\big|_{x=1}=\theta_{23}'\big|_{x=1}=0\,,\; w_{13}\big|_{x=\eta}=w_{23}\big|_{x=\eta}\,,\; \theta_{13}\big|_{x=\eta}=\theta_{23}\big|_{x=\eta}\,,\; \theta_{13}'\big|_{x=\eta}=\theta_{23}'\big|_{x=\eta}\,,$$

$$
\begin{aligned}
&\chi\,\nu\left(w_{13}'-\theta_{13}\right)\Big|_{x=\eta}-\chi\,\nu\left(w_{23}'-\theta_{23}\right)\Big|_{x=\eta}+\alpha\,D_0^2 w_{13}\Big|_{x=\eta}=e^{i\omega_1 T_0}\Bigg[-2\,\alpha\,i\,\omega_1\,\dot{A}\,Y_1\Big|_{x=\eta}-2\,\frac{1}{\nu}\,\bar{B}\,B\,A\left\{-\bar{\phi}_1'\;\phi_1'\;Y_1'+\bar{\phi}_2'\;\phi_2'\;Y_2'\right\}\Big|_{x=\eta}\\
&+\bar{A}\,A^2\left\{-\left\{\int_0^\eta \bar{Y}_1'\,Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\frac{1}{2}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\right\}\Bigg]+e^{-i\omega_1 T_0}\Bigg[+2\,\alpha\,i\,\omega_1\,\dot{\bar{A}}\,\bar{Y}_1\Big|_{x=\eta}\\
&-2\,\frac{1}{\nu}\,\bar{B}\,B\,\bar{A}\left\{-\bar{\phi}_1'\;\phi_1'\;\bar{Y}_1'+\bar{\phi}_2'\,.\phi_2'\;\bar{Y}_2'\right\}\Big|_{x=\eta}+\bar{A}^2\,A\left\{-\frac{1}{2}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\left\{\int_0^\eta \bar{Y}_1'\,Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\right\}\Bigg]\\
&-\frac{1}{2}\,A^3\,e^{3i\omega_1 T_0}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\frac{1}{2}\,\bar{A}^3\,\bar{A}\,e^{-3i\omega_1 T_0}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\\
&-\frac{1}{2}\,A^3\,e^{3i\omega_1 T_0}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\frac{1}{2}\,\bar{A}^3\,\bar{A}\,e^{-3i\omega_1 T_0}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\\
&-\frac{1}{\nu}\,B^2\,A\,e^{i(2\omega_2+\omega_1)T_0}\left\{-\phi_1'^2\,Y_1'+\phi_2'^2\,Y_2'\right\}\Big|_{x=\eta}-\frac{1}{\nu}\,\bar{B}^2\,\bar{A}\,e^{-i(2\omega_2+\omega_1)T_0}\left\{-\bar{\phi}_1'^2\,\bar{Y}_1'+\bar{\phi}_2'^2\,\bar{Y}_2'\right\}\Big|_{x=\eta}\\
&-\frac{1}{\nu}\,\bar{B}^2\,A\,e^{-i(2\omega_2-\omega_1)T_0}\left\{-\bar{\phi}_1'^2\,Y_1'+\bar{\phi}_2'^2\,Y_2'\right\}\Big|_{x=\eta}-\frac{1}{\nu}\,B^2\,\bar{A}\,e^{i(2\omega_2-\omega_1)T_0}\left\{-\phi_1'^2\,\bar{Y}_1'+\phi_2'^2\,\bar{Y}_2'\right\}\Big|_{x=\eta}
\end{aligned}
\tag{16}
$$

Solution to Eq.(16) at order ε^3 can be written as;

$$
w_{i3}(x,T_0,T_2)=\Psi_i(x,T_2)\,e^{i\,\omega_1 T_0}+W_i(x,T_2)+cc\,,\quad \theta_{i3}(x,T_0,T_2)=\varphi_i(x,T_2)\,e^{i\,\omega_2 T_0}+\Theta_i(x,T_2)+cc\,, \tag{17}
$$

where Ψ *and* φ are the functions for the secular terms, W and Θ are the functions for the non-secular terms and *cc* denotes complex conjugate of the preceding terms.

Taking excitation frequency as $\Omega=\omega_1+\varepsilon^2\,\sigma$ which in σ is defined detuning parameter of order *O(1)*, inserting expressions (17) into Eq.(16) and considering only the terms producing secularities, one has

$$
\begin{aligned}
&\chi\,\nu\left(\Psi_i''\,e^{i\omega_1 T_0}-\varphi_i'\,e^{i\omega_2 T_0}\right)+\omega_1^2\;\Psi_i\;e^{i\omega_1 T_0}=-\frac{1}{2}\,F_i\;e^{i\omega_1 T_0+i\sigma T_2}+e^{i\omega_1 T_0}\Bigg\{2\,i\,\omega_1\;\dot{A}\,Y_i+2\,i\,\omega_1\;\mu_i\;A\,Y_i+2\,\frac{1}{\nu}\;\bar{B}\;B\;A\left(\bar{\phi}_i'\,\phi_i'\,Y_i\right)'\\
&+A^2\;\bar{A}\left[-\left\{\int_0^\eta \bar{Y}_1'\,Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}Y_i''-\frac{1}{2}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\bar{Y}_i''\right]\Bigg\}+e^{-i\omega_1 T_0}\Bigg\{-2\,i\,\omega_1\;\dot{\bar{A}}\;\bar{Y}_i-2\,i\,\omega_1\;\mu_i\;\bar{A}\;\bar{Y}_i+2\,\frac{1}{\nu}\;\bar{B}\;B\;\bar{A}\left(\bar{\phi}_i'.\phi_i'.\bar{Y}_i\right)'\\
&+\bar{A}^2\;A\left[-\frac{1}{2}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}Y_i''-\left\{\int_0^\eta \bar{Y}_1'\,Y_1'\,dx+\int_\eta^1 \bar{Y}_2'\,Y_2'\,dx\right\}\bar{Y}_i''\right]\Bigg\}+\frac{1}{\nu}\;B^2\,A\,e^{i(\omega_1+2\omega_2)T_0}\left(\phi_i'^2\;Y_i'\right)'+\frac{1}{\nu}\;\bar{B}^2\;\bar{A}\,e^{-i(\omega_1+2\omega_2)T_0}\left(\bar{\phi}_i'^2.\bar{Y}_i'\right)'\\
&+\frac{1}{\nu}\;\bar{B}^2\,A\,e^{i(\omega_1-2\omega_2)T_0}\left(\bar{\phi}_i'^2\;Y_i'\right)'+\frac{1}{\nu}\;B^2\;\bar{A}\;e^{-i(\omega_1-2\omega_2)T_0}\left(\phi_i'^2\bar{Y}_i'\right)'-\frac{1}{2}\;A^2\;A\;e^{3i\omega_1 T_0}\left\{\int_0^\eta Y_1'^2\;dx+\int_\eta^1 Y_2'^2\;dx\right\}Y_i''\\
&-\frac{1}{2}\;\bar{A}^2\;\bar{A}\;e^{-3i\omega_1 T_0}\left\{\int_0^\eta \bar{Y}_1'^2\;dx+\int_\eta^1 \bar{Y}_2'^2\;dx\right\}\bar{Y}_i''
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_i''\,e^{i\omega_2 T_0}+\chi\;\nu\left(\Psi_i'\;e^{i\omega_1 T_0}-\varphi_i\;e^{i\omega_2 T_0}\right)+\omega_2^2\;\varphi_i\;e^{i\omega_2 T_0}=+e^{i\omega_2 T_0}\left\{2\;i\;\omega_2\;\dot{B}\;\phi_i+2\,\frac{1}{\nu}\;\bar{A}\;A\;B\left(\bar{Y}_i'\;Y_i'\;\phi_i'\right)'\right\}\\
&+e^{-i\omega_2 T_0}\left\{-2\;i\;\omega_2\;\dot{\bar{B}}\;\bar{\phi}_i+2\,\frac{1}{\nu}\;\bar{A}\;A\;\bar{B}\left(\bar{Y}_i'\,Y_i'\,\bar{\phi}_i'\right)'\right\}+\frac{1}{\nu}\;A^2\;B\;e^{i(2\omega_1+\omega_2)T_0}\left(Y_i'^2\;\phi_i'\right)'+\frac{1}{\nu}\;\bar{A}^2\;\bar{B}\;e^{-i(2\omega_1+\omega_2)T_0}\left(\bar{Y}_i'^2\;\bar{\phi}_i'\right)'\\
&+\frac{1}{\nu}\;\bar{A}^2\;B\;e^{-i(2\omega_1-\omega_2)T_0}\left(\bar{Y}_i'^2\;\phi_i'\right)'+\frac{1}{\nu}\;A^2\;\bar{B}\;e^{i(2\omega_1-\omega_2)T_0}\left(Y_i'^2\;\bar{\phi}_i'\right)'
\end{aligned}
$$

$$
\Psi_1\big|_{x=0}=\varphi_1'\big|_{x=0}=0\,,\quad \Psi_2\big|_{x=1}=\varphi_2'\big|_{x=1}=0\,,\quad \Psi_1\big|_{x=\eta}=\Psi_2\big|_{x=\eta}\,,\quad \varphi_1\big|_{x=\eta}=\varphi_2\big|_{x=\eta}\,,\quad \varphi_1'\big|_{x=\eta}=\varphi_2'\big|_{x=\eta}\quad ,
$$

$$\chi\,\nu\left(\Psi_1' e^{i\omega_1 T_0}-\varphi_1 e^{i\omega_2 T_0}\right)\Big|_{x=\eta}-\chi\,\nu\left(\Psi_2' e^{i\omega_1 T_0}-\varphi_2 e^{i\omega_2 T_0}\right)\Big|_{x=\eta}-\alpha\,\omega_1^2\,e^{i\omega_1 T_0}\Psi_1\big|_{x=\eta}=e^{i\omega_1 T_0}\Big[-2\,\alpha\,i\,\omega_1\,\dot{A}\,Y_1\big|_{x=\eta}$$
$$-2\frac{1}{\nu}\,\bar{B}\,B\,A\left\{-\bar{\phi}_1'\,\phi_1'\,Y_1'+\bar{\phi}_2'\,\phi_2'\,Y_2'\right\}\Big|_{x=\eta}+\bar{A}\,A^2\left\{-\left\{\int_0^\eta \bar{Y}_1' Y_1'\,dx+\int_\eta^1 \bar{Y}_2' Y_2'\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}\right.$$
$$\left.-\frac{1}{2}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\right\}\Big]+e^{-i\omega_1 T_0}\Big[+2\,\alpha\,i\,\omega_1\,\dot{\bar{A}}\,\bar{Y}_1\big|_{x=\eta}-2\frac{1}{\nu}\,\bar{B}\,B\,\bar{A}\left\{-\bar{\phi}_1'\,\phi_1'\,\bar{Y}_1'+\bar{\phi}_2'\,\phi_2'\,\bar{Y}_2'\right\}\Big|_{x=\eta}$$
$$+\bar{A}^2\,A\left\{-\frac{1}{2}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\left\{\int_0^\eta \bar{Y}_1' Y_1'\,dx+\int_\eta^1 \bar{Y}_2' Y_2'\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}\right\}\Big]$$
$$-\frac{1}{2}A^3 e^{3i\omega_1 T_0}\left\{\int_0^\eta Y_1'^2\,dx+\int_\eta^1 Y_2'^2\,dx\right\}\left\{Y_1'-Y_2'\right\}\Big|_{x=\eta}-\frac{1}{2}\bar{A}^3\,\bar{A}\,e^{-3i\omega_1 T_0}\left\{\int_0^\eta \bar{Y}_1'^2\,dx+\int_\eta^1 \bar{Y}_2'^2\,dx\right\}\left\{\bar{Y}_1'-\bar{Y}_2'\right\}\Big|_{x=\eta}$$
$$-\frac{1}{\nu}B^2 A\,e^{i(2\omega_2+\omega_1)T_0}\left\{-\phi_1'^2\,Y_1'+\phi_2'^2\,Y_2'\right\}\Big|_{x=\eta}-\frac{1}{\nu}\bar{B}^2\,\bar{A}\,e^{-i(2\omega_2+\omega_1)T_0}\left\{-\bar{\phi}_1'^2\,\bar{Y}_1'+\bar{\phi}_2'^2\,\bar{Y}_2'\right\}\Big|_{x=\eta}$$
$$-\frac{1}{\nu}\bar{B}^2 A\,e^{-i(2\omega_2-\omega_1)T_0}\left\{-\bar{\phi}_1'^2\,Y_1'+\bar{\phi}_2'^2\,Y_2'\right\}\Big|_{x=\eta}-\frac{1}{\nu}B^2\,\bar{A}\,e^{i(2\omega_2-\omega_1)T_0}\left\{-\phi_1'^2\,\bar{Y}_1'+\phi_2'^2\,\bar{Y}_2'\right\}\Big|_{x=\eta} \tag{18}$$

3.3. Using Symplectic Method

According to Symplectic Method, Eq.(15) can be converted into following form

$$A e^{i\omega_1 T_0} Y_i'=A e^{i\omega_1 T_0} Z_i\,,\quad B e^{i\omega_2 T_0}\phi_i'=B e^{i\omega_2 T_0}H_i$$
$$\chi\nu\left\{A e^{i\omega_1 T_0} Z_i'-B e^{i\omega_2 T_0}H_i\right\}+\omega_1^2 A e^{i\omega_1 T_0} Y_i=0\,,\ \chi\nu\left\{A e^{i\omega_1 T_0} Z_i-B e^{i\omega_2 T_0}\phi_i\right\}+B e^{i\omega_2 T_0}H_i'+\omega_2^2 B e^{i\omega_2 T_0}.\phi_i=0 \tag{19}$$

Eq.(19) gives the following matrix form of the problem.

$$\begin{bmatrix}\lambda & 0 & -1 & 0\\ 0 & \lambda & 0 & -1\\ \omega_1^2 & 0 & \chi\nu.\lambda & -\chi\nu\\ 0 & \left(-\chi\nu+\omega_2^2\right) & \chi\nu & \lambda\end{bmatrix}\begin{Bmatrix}A e^{i\omega_1 T_0} Y_i\\ B e^{i.\omega_2 T_0}\phi_i\\ A e^{i\omega_1 T_0} Z_i\\ B e^{i\omega_2 T_0}H_i\end{Bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{20}$$

Expressing Eq.(20) in the matrix form as $\frac{\partial X}{\partial x}=H\,X$ and choosing form $X=e^{\lambda x}$ for the solution gives us

$$\frac{\partial X}{\partial x}=\lambda X \tag{21}$$

where X*(x)* is the corresponding eigenvector and λ is the eigenvalue. From eigenvalue problem at Eq.(20), there are four eigenvalues. Thus, solution of the linear problem can be written as follows;

$$Y_i(x)=c_{i1}e^{\lambda_1 x}+c_{i2}e^{\lambda_2 x}+c_{i3}e^{\lambda_3 x}+c_{i4}e^{\lambda_4 x}\,,\quad \phi_i(x)=d_{i1}e^{\lambda_1 x}+d_{i2}e^{\lambda_2 x}+d_{i3}e^{\lambda_3 x}+d_{i4}e^{\lambda_4 x} \tag{22}$$

A solvability conditions must be satisfied for the non-homogenous equation in order to have a solution where the homogenous equation has a nontrivial solution [45,46]. For homogeneous problem, if the solution at order ε^3 is separated as secular and non-secular terms and the solvability condition is applied in Eq.(18) for eliminating secular terms, Eq.(18) can be converted to new symplectic form as follows:

$$e^{i\omega_1 .T_0}\Psi_i'-e^{i\omega_1 T_0}\beta_i=0\,,\ e^{i\omega_2 T_0}\varphi_i'-e^{i\omega_2 T_0}\gamma_i=0\,,\ \left[\chi\nu\,\beta_i'+\omega_1^2\,\Psi_i\right]e^{i\omega_1 T_0}-\chi\,\nu\,\gamma_i\,e^{i\omega_2 T_0}=0\,,$$
$$\left[\chi\nu\,\beta_i\right]e^{i\omega_1 T_0}+\left[\gamma_i'+\left(-\chi\nu+\omega_2^2\right)\varphi_i\right]e^{i\omega_2 T_0}=0$$
$$\Psi_1\big|_{x=0}=\gamma_1\big|_{x=0}=0\,,\quad \Psi_2\big|_{x=1}=\gamma_2\big|_{x=1}=0\,,\quad \Psi_1\big|_{x=\eta}=\Psi_2\big|_{x=\eta}\,,\ \varphi_1\big|_{x=\eta}=\varphi_2\big|_{x=\eta}\,,\ \gamma_1\big|_{x=\eta}=\gamma_2\big|_{x=\eta}\,,$$
$$\chi\,\nu\left\{\beta_1 e^{i\omega_1 T_0}-\varphi_1 e^{i\omega_2 T_0}\right\}\Big|_{x=\eta}-\chi\,\nu\left\{\beta_2 e^{i\omega_1 T_0}-\varphi_2 e^{i\omega_2 T_0}\right\}\Big|_{x=\eta}-\alpha\,\omega_1^2\,e^{i\omega_1 T_0}\Psi_1\big|_{x=\eta}=0 \tag{23}$$

If one invokes the solvability procedures given in Nayfeh and Mook[45] to these equations, the following trial functions can be obtained;

$$u_1=\chi\,\nu\left(B e^{i\omega_2 T_0}\phi_1-A e^{i\omega_1 T_0}Y_1'\right),\ u_2=-B e^{i\omega_2 T_0}\phi_1'-\chi\,\nu\,A e^{i\omega_1 T_0}Y_1\,,\ u_3=A e^{i\omega_1 T_0}Y_1\,,\ u_4=B e^{i\omega_2 T_0}\phi_1$$
$$u_5=\chi\,\nu\left(B e^{i\omega_2 T_0}\phi_2-A e^{i\omega_1 T_0}Y_2'\right),u_6=-B e^{i\omega_2 T_0}\phi_2'-\chi\nu\,A e^{i\omega_1 T_0}Y_2\,,\ u_7=A e^{i\omega_1 T_0}Y_2\,,\ u_8=B e^{i\omega_2 T_0}\phi_2 \tag{24}$$

Then, the trial functions can be used for non-homogeneous problem. After necessary calculations, one obtains the following equations;

$$2\, i\, \omega_2\, B\, D_2 B\, \bar{d} + \frac{1}{\nu} B^2\, A\, \bar{A} \left[\left\{ -\phi_1'^2\, \bar{Y}_1' + \phi_2'^2\, \bar{Y}_2' \right\} \Big|_{x=\eta} Y_1 \big|_{x=\eta} + \left\{ \int_0^\eta \left(\phi_1'^2\, \bar{Y}_1' \right)' Y_1\, dx + \int_\eta^1 \left(\phi_2'^2\, \bar{Y}_2' \right)' Y_2\, dx \right\} + 2.\left\{ \int_0^\eta \left(\bar{Y}_1'\, Y_1'\, \phi_1' \right)' \phi_1\, dx + \int_\eta^1 \left(\bar{Y}_2'\, Y_2'\, \phi_2' \right)' \phi_2\, dx \right\} \right] = 0$$

$$2\, i\, \omega_1\, m\, A\, D_2 A + 2\, i\, \omega_1\, \mu\, A^2 - \frac{1}{2} A\, f\, e^{i \sigma T_2} + A^3\, \bar{A} \left[\left\{ \left\{ \int_0^\eta \bar{Y}_1' Y_1'\, dx + \int_\eta^1 \bar{Y}_2' Y_2'\, dx \right\} \{ Y_1' - Y_2' \} \big|_{x=\eta} \right\} \right.$$

$$+ \frac{1}{2} \left\{ \int_0^\eta Y_1'^2\, dx + \int_\eta^1 Y_2'^2\, dx \right\} \{ \bar{Y}_1' - \bar{Y}_2' \} \big|_{x=\eta} \Bigg\} Y_1 \big|_{x=\eta} - \left\{ \int_0^\eta \bar{Y}_1' Y_1'\, dx + \int_\eta^1 \bar{Y}_2' Y_2'\, dx \right\} \left\{ \int_0^\eta Y_1'' Y_1\, dx + \int_\eta^1 Y_2'' Y_2\, dx \right\}$$

$$- \frac{1}{2} \left\{ \int_0^\eta Y_1'^2\, dx + \int_\eta^1 Y_2'^2\, dx \right\} \left\{ \int_0^\eta \bar{Y}_1'' Y_1\, dx + \int_\eta^1 \bar{Y}_2'' Y_2\, dx \right\} \Bigg] + \frac{1}{\nu} \bar{B}\, B\, A^2 \left[2 \left\{ -\bar{\phi}_1'\, \phi_1'\, Y_1' + \bar{\phi}_2'\, \phi_2'\, Y_2' \right\} \Big|_{x=\eta} Y_1 \big|_{x=\eta} \right.$$

$$\left. + 2.\left\{ \int_0^\eta \left(\bar{\phi}_1' \phi_1' Y_1' \right)' Y_1\, dx + \int_\eta^1 \left(\bar{\phi}_2' \phi_2' Y_2' \right)' Y_2\, dx \right\} + \left\{ \int_0^\eta \left(Y_1'^2\, \bar{\phi}_1' \right)' \phi_1\, dx + \int_\eta^1 \left(Y_2'^2\, \bar{\phi}_2' \right)' \phi_2\, dx \right\} \right] = 0$$

$$\int_0^\eta Y_1^2\, dx + \int_\eta^1 Y_2^2\, dx = 1\,, \quad f = \int_0^\eta F_1 Y_1\, dx + \int_\eta^1 F_2 Y_2\, dx\,, \quad \mu_1 = \mu_2 = \mu\,, \quad \hat{d} = \int_0^\eta \hat{\phi}_1^2\, dx + \int_\eta^1 \hat{\phi}_2^2\, dx\,, \; m = 1 + \alpha\, Y_1^2 \big|_{x=\eta}\,. \tag{25}$$

Thus, after simplifications on terms having (^) are described as follow for numerical analysis,

$$\bar{\phi}\phi = \frac{\bar{A}\, e^{-i \omega_1 T_0}}{\bar{B}\, e^{-i \omega_2 T_0}} \frac{A e^{i \omega_1 T_0}}{B e^{i \omega_2 T_0}} = \frac{\bar{A} A}{\bar{B} B} \bar{\hat{\phi}}\, \hat{\phi}\,, \quad \phi^2 = \frac{A e^{i \omega_1 T_0}}{B\, e^{i \omega_2 T_0}} \frac{A\, e^{i \omega_1 T_0}}{B\, e^{i \omega_2 T_0}} = \frac{A^2 e^{2 i \omega_1 T_0}}{B^2 e^{2 i \omega_2 T_0}} \hat{\phi}^2 \tag{26}$$

Eq.(25) can be written as follows;

$$2 i \omega_1 m\, D_2 A + 2 i \omega_1 \mu A - \frac{1}{2} f\, e^{i \sigma T_2} + \bar{A}\, A^2\, \Delta_1 + A \bar{B}\, B \frac{1}{\nu} \Delta_2 = 0\,, \qquad A\, \bar{A}\, B \frac{1}{\nu}\, \Delta_3 + 2\, i\, \omega_2\, D_2 B\, \hat{d} = 0 \tag{27}$$

Simplification yields following formation;

$$2\, i\, \omega_1\, m\, \dot{A} + 2\, i\, \omega_1\, \mu\, A + \left(\Delta_1 + \frac{1}{\nu} \Delta_2 \right) \bar{A}\, A^2 = \frac{1}{2} f\, e^{i \sigma T_2}\,, \quad A \bar{A}\, B \frac{1}{\nu} \Delta_3 + 2 i \omega_2\, \dot{B}\, \hat{d} = 0$$

$$\Delta_1 = -\left[-\left\{ \int_0^\eta \bar{Y}_1' Y_1'\, dx + \int_\eta^1 \bar{Y}_2' Y_2'\, dx \right\} \{ Y_1' - Y_2' \} \big|_{x=\eta} - \frac{1}{2} \left\{ \int_0^\eta Y_1'^2\, dx + \int_\eta^1 Y_2'^2\, dx \right\} \{ \bar{Y}_1' - \bar{Y}_2' \} \big|_{x=\eta} \right] Y_1 \big|_{x=\eta}$$

$$- \left\{ \int_0^\eta \bar{Y}_1' Y_1' dx + \int_\eta^1 \bar{Y}_2' Y_2' dx \right\} \left\{ \int_0^\eta Y_1'' Y_1\, dx + \int_\eta^1 Y_2'' Y_2\, dx \right\} - \frac{1}{2} \left\{ \int_0^\eta Y_1'^2\, dx + \int_\eta^1 Y_2'^2\, dx \right\} \left\{ \int_0^\eta \bar{Y}_1'' Y_1\, dx + \int_\eta^1 \bar{Y}_2'' Y_2\, dx \right\}$$

$$\Delta_2 = 2 \left\{ -\bar{\hat{\phi}}_1'\, \hat{\phi}_1'\, Y_1' + \bar{\hat{\phi}}_2'\, \hat{\phi}_2'\, Y_2' \right\} \Big|_{x=\eta} Y_1 \big|_{x=\eta} + 2 \left\{ \int_0^\eta \left(\bar{\hat{\phi}}_1'\, \hat{\phi}_1'\, Y_1' \right)' Y_1\, dx + \int_\eta^1 \left(\bar{\hat{\phi}}_2'\, \hat{\phi}_2' Y_2' \right)' Y_2\, dx \right\} + \left\{ \int_0^\eta \left(Y_1'^2\, \bar{\hat{\phi}}_1' \right)' \hat{\phi}_1\, dx + \int_\eta^1 \left(Y_2'^2\, \bar{\hat{\phi}}_2' \right)' \hat{\phi}_2\, dx \right\}$$

$$\Delta_3 = \left\{ -\hat{\phi}_1'^2\, \bar{Y}_1' + \hat{\phi}_2'^2\, \bar{Y}_2' \right\} \Big|_{x=\eta} Y_1 \big|_{x=\eta} + \left\{ \int_0^\eta \left(\hat{\phi}_1'^2\, \bar{Y}_1' \right)' Y_1\, dx + \int_\eta^1 \left(\hat{\phi}_2'^2\, \bar{Y}_2' \right)' Y_2\, dx \right\} + 2 \left\{ \int_0^\eta \left(\bar{Y}_1' Y_1'\, \hat{\phi}_1' \right)' \hat{\phi}_1\, dx + \int_\eta^1 \left(\bar{Y}_2' Y_2'\, \hat{\phi}_2' \right)' \hat{\phi}_2\, dx \right\} \tag{28}$$

Complex amplitude *A* and *B* can be written in terms of real amplitudes *a* and *b*, and phases ς_1 and ς_2

$$A(T_2) = \frac{1}{2}\, a(T_2)\, e^{i \varsigma_1 (T_2)}, \qquad B(T_2) = \frac{1}{2}\, b(T_2)\, e^{i \varsigma_2 (T_2)} \tag{29}$$

Substituting Eq.(29) into Eq.(28), and separating real and imaginary parts, following amplitude-phase modulation equations can be finally obtained;

$$\omega_1\, m\, \dot{a} + \omega_1\, \mu\, a = \frac{1}{4} f\, \sin\tau, \quad -\omega_1\, m\, a\, (\sigma - \dot{\tau}) + \frac{1}{8} \left(\Delta_1 + \frac{1}{\nu} \Delta_2 \right) a^3 = \frac{1}{4} f\, \cos\tau\,, \; \omega_2\, \hat{d}\, \dot{b} = 0, \; -\omega_2\, \hat{d}\, b\, \dot{\varsigma}_2 + \frac{1}{8} a^2\, b\, \frac{1}{\nu} \Delta_3 = 0 \tag{30}$$

where τ is defined as

$$\tau = \sigma\, T_2 - \varsigma_1 \tag{31}$$

IV. NUMERICAL RESULTS

4.1 Solutions to the Linear Problem; Natural Frequencies

Table 1 First five natural frequencies in transversal directions for different mass locations and mass ratios.

			n=0.01					***n=1.0***				
ν	α	η	$(\omega_1)_1$	$(\omega_1)_2$	$(\omega_1)_3$	$(\omega_1)_4$	$(\omega_1)_5$	$(\omega_1)_1$	$(\omega_1)_2$	$(\omega_1)_3$	$(\omega_1)_4$	$(\omega_1)_5$
100	***0.1***	***0.1***	8.5444	25.4252	42.4182	59.4685	77.8985	2.9868	6.1949	9.3400	12.4650	15.5911
		0.2	8.3353	24.2319	42.6038	63.2669	83.6472	2.9796	6.1826	9.3298	12.5035	15.6752
		0.3	8.1005	24.4467	45.3470	62.4807	77.2918	2.9706	6.1827	9.3662	12.5031	15.5945
		0.4	7.9297	25.6206	44.2219	60.2674	83.6472	2.9636	6.1950	9.3549	12.4682	15.6753

		0.5	7.8686	26.4159	41.8846	64.8496	76.9886	2.9609	6.2027	9.3257	12.5256	15.5952
	1	***0.1***	7.7659	17.8116	33.0022	53.2161	73.8996	2.9614	6.1226	9.0739	12.0325	15.1311
		0.2	6.3583	17.3434	37.8566	60.9826	83.6472	2.8891	6.0131	9.0657	12.3877	15.6752
		0.3	5.4820	19.5463	44.1712	55.3504	71.2801	2.8065	6.0241	9.3380	12.3697	15.2315
		0.4	5.0422	23.1037	39.4357	53.8981	83.6472	2.7458	6.1319	9.2488	12.1761	15.6752
		0.5	4.9087	26.4160	32.9967	64.8496	67.6898	2.7240	6.2027	9.0389	12.5255	15.2508
	10	***0.1***	4.0025	12.1526	31.3779	52.4515	73.4090	2.6752	5.2022	7.7876	11.1475	14.5941
		0.2	2.6259	14.5620	36.8107	60.4257	83.6472	2.1464	4.9381	8.4969	12.2360	15.6753
		0.3	2.1133	17.6704	43.5870	53.3905	70.5681	1.8290	5.2413	9.2754	12.1440	14.8707
		0.4	1.8924	21.8393	37.6782	52.7989	83.6472	1.6732	5.7499	8.9876	11.8026	15.6753
		0.5	1.8288	26.4160	29.9024	64.8496	65.8654	1.6266	6.2027	8.3534	12.5256	14.9226
10000	***0.1***	***0.1***	9.7563	37.8414	82.0320	142.1523	220.0498	2.9909	6.1998	9.3653	12.5197	15.6700
		0.2	9.5219	36.1283	81.6273	149.6268	235.2649	2.9841	6.1913	9.3629	12.5240	15.6762
		0.3	9.2563	36.3592	86.6157	148.7995	219.7275	2.9756	6.1914	9.3712	12.5241	15.6700
		0.4	9.0607	38.0507	84.8026	143.4769	235.2649	2.9689	6.1998	9.3686	12.5197	15.6762
		0.5	8.9902	39.1621	80.6498	153.0670	219.5412	2.9663	6.2050	9.3619	12.5267	15.6700
	1	***0.1***	8.9774	29.6046	64.8077	123.3281	203.6078	2.9679	6.1539	9.3063	12.4597	15.6181
		0.2	7.4384	26.7286	72.2423	144.8036	235.2649	2.9017	6.0788	9.2905	12.5042	15.6764
		0.3	6.3820	29.5184	85.1838	138.5852	199.5854	2.8260	6.0880	9.3638	12.5034	15.6224
		0.4	5.8358	34.9526	78.5128	128.5631	235.2649	2.7704	6.1615	9.3420	12.4677	15.6762
		0.5	5.6691	39.1621	66.6606	153.0670	196.8681	2.7503	6.2051	9.2859	12.5267	15.6239
	10	***0.1***	5.3120	19.6196	58.0257	118.9895	200.5257	2.7459	5.7696	8.8975	12.0958	15.3284
		0.2	3.2510	21.8794	69.5544	143.5071	235.2649	2.2972	5.5404	9.0105	12.4240	15.6762
		0.3	2.5224	26.5618	84.5993	134.6713	195.0857	1.9834	5.7310	9.3388	12.4257	15.4493
		0.4	2.2209	33.4018	75.8119	124.5349	235.2649	1.8195	6.0400	9.2523	12.2975	15.6762
		0.5	2.1355	39.1621	61.2882	153.0670	190.8995	1.7692	6.2051	9.0380	12.5267	15.4722

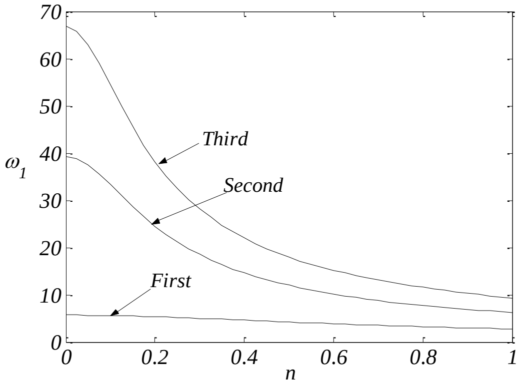

Figure 2. Natural frequency - frequency ratio for three modes of the beam.

In numerical analysis, according to eigenvalue problem solutions ($\mp\bar{\lambda}.i\ ,\mp\wp$) can be rewritten in the following form:

$$Y_i(x)=c_{i1}\cos(\bar{\lambda}x)+c_{i2}\sin(\bar{\lambda}x)+c_{i3}\cosh(\wp x)+c_{i4}\sinh(\wp x),$$
$$\phi_i(x)=d_{i1}\cos(\bar{\lambda}x)+d_{i2}\sin(\bar{\lambda}x)+d_{i3}\cosh(\wp x)+d_{i4}\sinh(\wp x) \qquad (32)$$

Inserting these forms into Eq.(20), one can obtain the following solutions:

$$Y_i(x)=c_{i1}\cosh(\bar{\lambda}x)+c_{i2}\sinh(\bar{\lambda}x)+c_{i3}\cos(\wp x)+c_{i4}\sin(\wp x)$$
$$\phi_i(x)=\frac{A\,e^{i\omega_1 T_0}}{B\,e^{i\omega_2 T_0}}\cdot\left\{c_{i1}\frac{\{-\chi\,\nu\,\bar{\lambda}^2+\omega_1^2\}}{\chi\,\nu\,\bar{\lambda}}\sin(\bar{\lambda}\,x)+c_{i2}\frac{\{-\chi\,\nu\,\bar{\lambda}^2+\omega_1^2\}}{-\chi\,\nu\,\bar{\lambda}}\cos(\bar{\lambda}\,x)+c_{i3}\frac{\{\chi\,\nu\,\wp^2+\omega_1^2\}}{\chi\,\nu\,\wp}\sinh(\wp\,x)+c_{i4}\frac{\{\chi\,\nu\,\wp^2+\omega_1^2\}}{\chi\,\nu\,\wp}\cosh(\wp\,x)\right\} \qquad (33)$$

In our study for natural frequencies, assumption of $\omega_2=n\,\omega_1$ has been done so that *n* can be defined as a ratio of rotational frequency to transverse frequency. After obtaining eigenvalues from Eq.(20) and using solution function at Eq.(32), transverse natural frequencies(ω_1) can be calculated via conditions at Eq.(15). In numerical studies, material properties were considered as constant due to slenderness ratio(ν), and shear/flexural rigidity ratio(χ) or Poisson's ratio(υ) were investigated in detail. Throughout numerical calculations, Poisson's ratio and shear correction coefficient are assumed 0.30 and k=5/6, respectively.

Using the slenderness ratio(ν=10000), mass ratio(α=1), mass location (η=0.5), one can plot ω_1 versus *n* graphs for first three modes as seen in Fig.2. At this figure, increasing frequency ratio decreases linear natural frequencies. At 0<*n*<0.5, this decreasing is the fastest for the second mode's natural frequency.

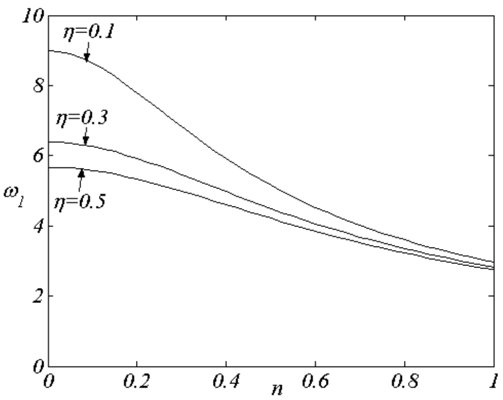

Figure 3. Natural transverse frequencies via frequency ratio for different mass locations, α=1, ν =10000.

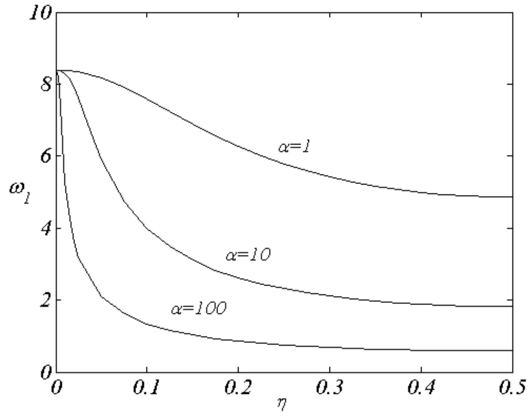

Figure 4. Natural transverse frequencies via mass location for different mass ratios,n=0.1, ν =100.

In Fig.3, change of transverse frequency versus frequency ratio has been investigated for first mode vibration of the beam. If the mass is kept near the ends of the beam, high natural frequencies can be obtained. Increasing the frequency ratio decreases natural frequencies. Selecting slenderness ratio as 100, change of transverse frequency versus mass location has been investigated for first mode vibration of the beam in Fig. 4. Increasing the mass magnitude decreases natural frequencies.

The first five natural frequencies are given for *n*=0.01 and *n*=1.00 at Table 1. From these values, increasing mass location and mass ratio resulted in decreasing natural frequencies only for the first mode. Other modes' natural frequencies are very complex. If one makes comparison between both tables, lower frequency ratios having higher natural frequencies can be seen. This means that energy of the system is transferred to rotation of the beam. This means that energy is divided equally between rotational and transverse vibration modes in case of *n*=1.

4.1 Solutions to the Non-Linear Problem; Force-response curves

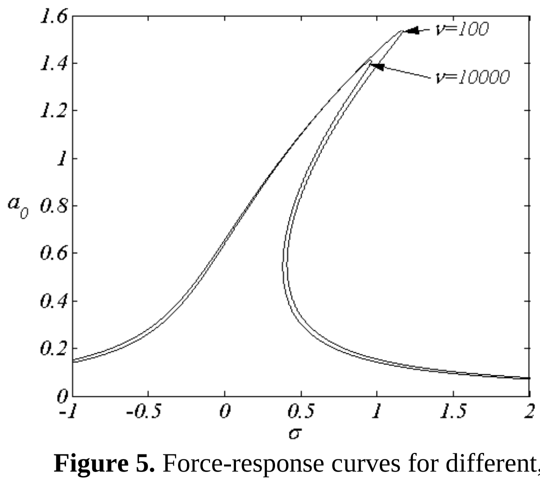

Figure 5. Force-response curves for different, slenderness ratios n=1.0, η=0.5, α=1.

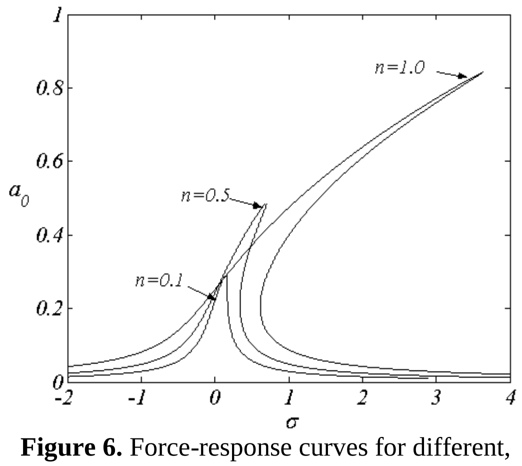

Figure 6. Force-response curves for different, frequency ratio η=0.5, α=0.1, ν=10000.

For steady state in Eqs.(30), amplitudes vanish with increasing time. This is in brief;

$$\dot{a}=0 \,\&\, \dot{b}=0 \quad \rightarrow \quad a=a_0 \,\&\, b=b_0 \ \text{(Constant)} \tag{34}$$

Note that a_0 and b_0 are the steady state real amplitudes of the response.

Using Eq.(30), one obtains following equations;

$$\omega_1\ \mu\ a_0=\frac{1}{2}\ f\ \sin\tau, \qquad -\omega_1\ m\ a_0\ (\sigma-\dot{\tau})+\frac{1}{8}\left(\Delta_1+\frac{1}{\nu}.\Delta_2\right)\ a_0^3=\frac{1}{2}\ f\ \cos\tau \tag{35}$$

After some manipulations for steady state case, we obtain following detuning parameter;

$$\sigma = \aleph\, a_0^2 \pm \sqrt{\left(\frac{\tilde{f}}{2\,\omega_1\, a_0}\right)^2 - \tilde{\mu}^2}\,,\ \aleph = \frac{1}{8}\left(\Delta_1 + \frac{1}{\nu}\,\Delta_2\right)\frac{1}{\omega_1\, m}\,,\ \tilde{\mu} = \frac{\mu}{m}\,,\ \ \tilde{f} = \frac{f}{m} \tag{36}$$

Here, $\aleph$ has been described as frequency correction coefficient to linear frequency.

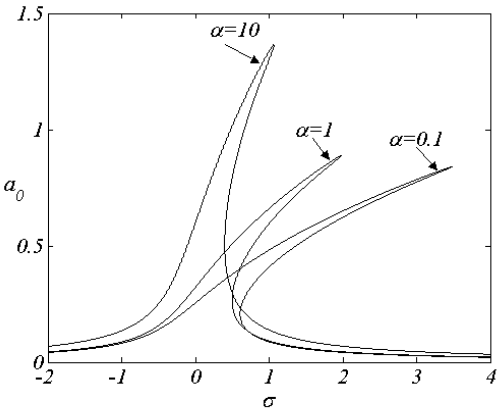

Figure 7. Force-response curves for different mass ratio,n=1, η=0.3, ν=100.

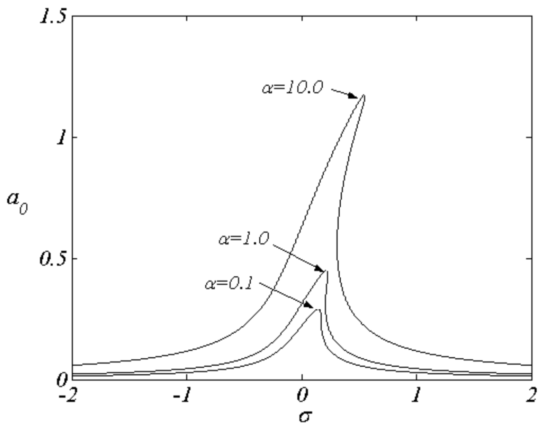

Figure 8. Force-response curves for different mass ratio,n=0.1, η=0.5, ν=10000.

Using Eq.(36), frequency–response graphs were drawn at Figs.5-8. $\tilde{f}$ =1 and $\tilde{\mu}$ =0.1 were taken at these graphs. Frequency–response graphs via different slenderness ratios were given in Fig.5 for n=1.0, η=0.5, and α=1. Graph shows that decreasing ν increases hardening type behavior. Maximum amplitude value increases while ν increases and jump region gets greater.

Frequency–response graphs via different frequency ratios were given in Fig.6 for α=0.1, η=0.5, ν=10000. Graph shows that increasing frequency ratio (n) increases hardening type behavior. Maximum amplitude value increases while n increases and jump region gets greater. Another thing seen from this graph is the hardening behavior is less in case of lower frequency ratio.

Frequency–response graphs via different concentrated mass magnitudes were given in Figs. 7-8. In Fig.7, graph shows that increasing mass ratio decreases hardening type behavior. Maximum amplitude value increases while the mass ratio increases at a certain location. In Fig.8, graph shows increasing mass ratio makes jump region wider.

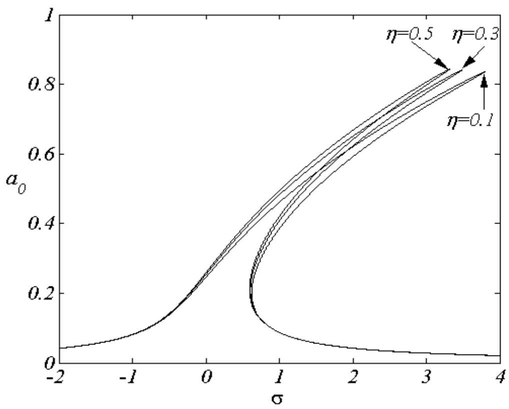

Figure 9. Force-response curves for different mass locations, n=1, α=0.1, ν=100.

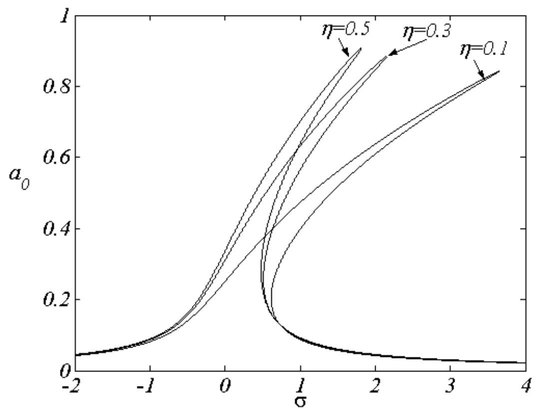

Figure 10. Force-response curves for different mass locations, n=1, α=1, ν=10000.

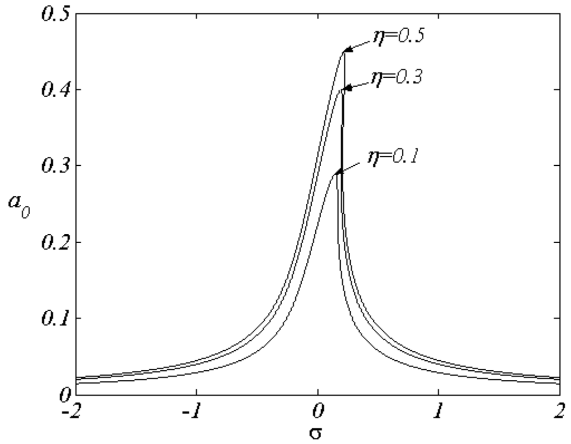

Figure 11. Force-response curves for different mass locations, n=0.1, α=1, v=10000.

Frequency–response graphs via different concentrated mass locations were given in Figures. 9-11. In Fig. 9, graph shows moving mass location from end points to the middle point of the beam causes hardening type behavior. For different frequency ratio (n=1) in Fig. 10, moving mass location makes less hardening behavior and small rising in maximum amplitude values. Changing some control parameters, our results become compatible with the studies of Pakdemirli et.al [40] and Özkaya et.al [38] which are based on Euler type beam. In Fig. 11, maximum amplitude value increases while concentrated mass gets closer to the midpoint and jump region gets wider.

V. CONCLUSIONS

In this study, nonlinear vibrations were investigated for the Timoshenko type beams carrying concentrated mass. For that purpose, equation of motions has been derived by using by using Hamilton Principle. To solve this coupled differential equations analytically Method of Multiple Scales (a perturbation method) has been used. The problem has been defined with solution orders; linear problem and non-linear problem. Solutions of the linear problem correspond to the natural frequencies. Assuming a ratio between rotational mode frequency and transversal mode frequency and defining this ratio as the frequency ratio, natural frequencies has been obtained by using different control parameters: location and magnitude of the concentrated mass, slenderness and frequency ratio. Natural frequencies decrease with increasing frequency ratio (n). Increasing frequency ratio resulted in sharing energy of the system between rotational and transversal modes. Holding mass up close to middle location of the beam would result in decreasing natural frequencies. And natural frequencies decrease with increasing the mass magnitude. Solutions of the non-linear problem correspond to forced vibration results, and were obtained by means frequency response curves in the case of steady-state of the system. Replacing concentrated mass to middle point of the beam instead of end points, would result in expanding multi-valued region, but would not change maximum amplitudes of vibrations for Timoshenko type beams. Using different slenderness and frequency ratio, one can obtain Euler-Bernoulli results; multi-valued region doesn't expand, maximum amplitudes of vibrations become larger. For low magnitude of the mass multi-valued regions are wide, but maximum amplitudes of vibrations are small, but for great magnitude of the mass the multi-valued regions are narrow, but maximum amplitudes of vibrations are larger. Frequency ratio causes hardening type behavior on the system. Thus, multi-valued region expands, maximum amplitudes of vibrations become larger as these parameters increase. When compared wıth Euler Bernoulli type beams generally speaking, Timoshenko type beams have hardening behavior, wide multi valued regions and larger maximum amplitudes of vibrations. As a future work nonlinear vibrations of Timoshenko type moving continua with any attachments (spring, mass) could be analyzed.
This study could be seen as a key stone to study axially moving Timoshenko beams, because problem using Euler-Bernoulli beam theory has been investigated by Sarıgül and Boyacı [37]. In case of carrying multiple concentrated masses, vibrations of plate using Timoshenko beam theory could be investigated.

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