REALIZATION OF FRACTIONAL ORDER DIFFERENTIATOR BY ANALOGUE ELECTRONIC CIRCUIT

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ABSTRACT

The aim of this paper is to realize fractional order derivative (FD) with electronic circuit utilizing the syntheses of a fractance circuit. The realization has been done via synthesis of two type of fractances as passive RC circuits connected in operational amplifier. The continuous fractional expansion (CFE) and partial fraction expansion (PE) have been utilized on the (FD) transfer function which has been approximated to integer order rational transfer function depending on Carlson, Caurer and Foster method through special Matlab instructions. As case of study FD raised to order 0.9 has been implemented with electronic realization. The analogue electronic circuit has been simulated using Circuit Wizard to realize the derivative function. The time, frequency domain response has been tested for both real fractional order derivative and the approximated one

KEYWORD: *Fractional order derivative (FD) , Nintegerv.2.3Fractional control toolbox , Partial fraction expansion (PE) , Continuous fractional expansion (CFE),fractance.*

I.   INTRODUCTION

The derivative and integral of a function is major mathematical tool in fractional and non fractional calculus for continues time and discrete time system. The fractional calculus is recently a new trend to be applied in most update engineering and other science branches especially automatic control system .The fractional proportional, integral, derivative (FOPID) controller which include a fractional derivative term when the derivative defined as:-

$$(1) D_a^\alpha u(t) = \begin{cases} \frac{d^\alpha}{dt^\alpha} & u(t) & if < 1 \\ \frac{d}{dt} & u(t) & if \ \alpha = 0 \end{cases}$$

In order to implement the derivative term practically as analog electronic circuit utilizing synthesis method, the fractional order derivative approximated to integer order rational transfer function

$$(2) G(s) = \frac{Q^n(s)}{P^n(s)}$$

Where $Q^n(s)$, $P^n(s)$ are polynomial of order n and the Approximation take into account selected frequency band (wl,wh) .The equivalent approximated transfer function G(s)can be synthesized as passive (RC) circuit and active electronic circuit using continues fractional expansion (CFE) and partial expansion (PE) for G(s).
In Laplace form the derivative function given as:

$$(3) U(s) = kd \ s \ U(s)$$

 With (kd) known as differentiator constant , The realization of it as analog circuit gives in fig (1)
Where the output give as:

$$U(t) = -RC \frac{de(t)}{dt} \ e(t) \qquad (4)$$

The parameter (Kd) become equal RC and the negative
Output can be eliminated by inverting circuit.

In the fractional order differentiator which the order become fraction or no integer value given as:

$$u(t) = kd \; \frac{d^\lambda}{dt^\lambda} \; U(s) \qquad (5)$$

The fractional derivative is one of three term of (FOPID) has the construction as in fig (3) which has fife parameters to be tuned in the implementation and its expression is:-

u(t)=k_pe(t)+$k_{i\;0}$D$^{-\lambda}_t$e(t)+$k_{d\;0}$D$^\lambda_t$ e(t)   (6)
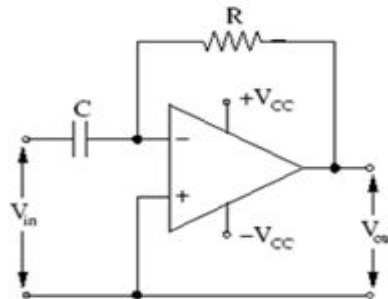

Figure (1): Derivative Type

$$vo = -Rc \; \frac{dvi}{dt} \qquad (7)$$

The realization of it as analog circuit using operational amplifier like as the integer order circuit with additional circuit called fractance (Z_f), fig (2)
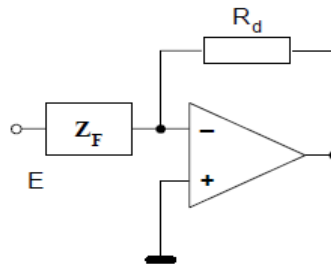

Figure (2): Analog Fractional Order Differentiator

The circuit that exhibiting and performing fractional-order behavior is called a fractance [1] in electronic circuit .Design of fractance can be done easily using any of the aforementioned rational approximations or a truncated CFE, which also gives a rational approximation. Truncated CFE does not require any further transformation rational approximation. The values of the electric elements, which are necessary for building a fractance, are then determined from the obtained finite continued fraction. If all coefficients of the obtained finite continued Fraction is positive, then the fractance can be made of classical passive elements (resistors and capacitors). If some of the coefficients are negative, then the fractance can be made with the help of negative impedance converters [2].
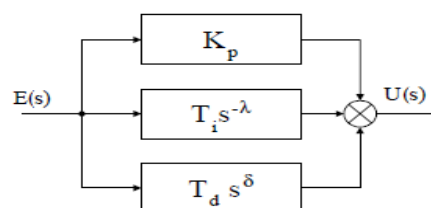

Figure (3): General FOPID Structure

The realization of FOPID taken into the count by many authors like Podlubny and I.Petras (2001) which the method of realization has been presented for fractional order system and synthesis of electronic circuit with realization[2] .later author B.T.Krishna and K.V.S.Reddy (2008) pointed to active and passive realization of fractional order with (1/2)[3]. The author G. Sridhar and K. Hrishikes(2012)[4] addressed design and to achieve dominant fractional by simulink using MATLAB and give a comparison of results between the proposed with others in which they found the dominant fractional is the best .Then Lubomir Dorcak and Juraj Valsa (2013)[5] where pointed to analogue realization to FO dynamical system. In this paper the method of Carlosn has been used for the approximation of the FD and continues fractional expansion (CFE) has been implemented for fractional derivative raised to order (0.9). Paper organized content as follows: Section (2) taking the study of the most important definitions. Section (3)-(3.3) showing the approximations for fractional order with different method .Section 4-(4.2) represent analog circuit by used two method Cure and Foster. Section(5) Simulation the electronic circuit by designing program called Circuit Wizard . Section (5) content the conclusion for this paper. Sections (6) represent future work.

II.   MATHEMATICAL BACK GROUND

There are two main approaches for defining fractional derivative. The first considers differentiation and integration as limits of finite differences. The Grunwald-Letnikov definition follows this approach. The other approach generalizes a convolution type representation of repeated integration [5]. Riemann-Liouville and Caputo fractional derivatives are fundamentally related to fractional integration operators. Consequently, the initial conditions of fractional derivatives are the frequency distributed and infinite dimensional state vector of fractional integrators.
 for higher order derivatives and   The Grunwald-Letnikov approach presents limit definitions integrals and shown that.

$$aD^\alpha t f(t) = \lim_{h \to 0} \frac{1}{(h)}^\alpha \sum_{k \to 0}^{\frac{t}{h}} (-1)^k \binom{a}{k} f(t - \alpha h) \qquad (8)$$

Where h is the time increment.
 While the Riemann-Louville definition is:-

$$aD^\alpha t f(t) = \frac{1}{\Gamma(n-\alpha)} d^n/(dt)^n \int_a^t \left(\frac{f(\tau)d(\tau)}{(t-\tau)^{\alpha-n+1}} \right) \qquad (9)$$

$$n - 1 \le \alpha < n$$

Finally the Caputo definition of fractional differentiation of fractional order α, can be written as

$$D^\alpha . f(t) = f^{m-\alpha} D^m f(t) \qquad (10)$$

$$\text{with}$$

$$m - 1 < \alpha \le m$$

$$D^\alpha . f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t f^m(\tau), m - 1 < \alpha < m \\[2ex] \dfrac{d^m}{dt^m} f(t), \alpha = m \end{cases} \qquad (11)$$

III.   FRACTIONAL ORDER APPROXIMATION

Several method used to obtain the approximated rational fractional transfer function with integer order. These method given by   Crone, Carlson, Matsuda, Ousta loop [6] which most usably, each method has its own properties. Utilizing MatlabRb2012 program and setting the special tool (Ninteger tool box)[7]

3.1 Carlson's method

The method proposed by Carlson in [6], derived from a regular Newton process used for iterative approximation of the α-th root, can be considered as belonging to this group. The starting point of the method is the statement of the following relationships:

$$(H(s))1/\alpha - (G(s)) = 0; H(s) = (G(s))\alpha$$

Defining α = 1/q, m = q/2, in each iteration, starting from the initial value H0(s) = 1, an approximated rational function is obtained in the form

$$Hi = Hi - 1(s)$$
$$\left.\begin{array}{c} (q - m)\big(Hi - 1(s)\big)2 + (q + m)G(s) \\ (q + m)\big(Hi - 1(s)\big)2 + (q - m)G(s) \end{array}\right\} \qquad (12)$$

The application of Carlson's method by implemented Ninteger tool box in Matlab given Fig(4). The fractional derivative of($S^{0.9}$) has been approximated by using Carlson's method which is given below where parameter taken as n=5,frequency=[0.01:100] ,α=0.9,k=1.

```
                FILE                              EDIT
  carls.m*   ✕   +
1 -     clc
2 -     clear
3 -     k=1;
4 -     a=10/9;
5 -     w=1;
6 -     N=5;
7 -     G = carlson(k, a, w, N);
8 -     f=minreal(G)
```

Figure (4) :Mat lab Screen m.file

The result appears as transfer function in command window given Fig (5).

```
G =

  9 s^6 + 138 s^5 + 711 s^4 + 1548 s^3 + 1351 s^2 + 330 s + 9
  ----------------------------------------------------------
    s^6 + 42 s^5 + 351 s^4 + 1164 s^3 + 1647 s^2 + 810 s + 81

Continuous-time transfer function.


f =

  9 s^4 + 84 s^3 + 126 s^2 + 36 s + 1
  -----------------------------------
    s^4 + 36 s^3 + 126 s^2 + 84 s + 9

Continuous-time transfer function.
```

Figure (5): Result Carlson Approximation In Command Window In Matlab

.And the same think there are Matlab instruction for other methods [7] given in section (3.2),(3.3).

3.2. Crone method

For: $H(s) = s^{\mu}, \mu \in R$

 The expansion given as:

$$H(s) = C \prod_{k=-N}^{N} \frac{1 + \frac{s}{wk}}{1 + \frac{s}{w'k}} \qquad (13)$$

The mat lab instruction is:

 C = crone1   (k, v, wl, wh, n)

3.3. Matsuda method

the expansion given as :

$$(14) H(s) = ao + \frac{s - so}{a1+} \frac{s - s1}{a2+} \frac{s - s2}{a3+}$$

 $A_i = v_i(s_i)$, $v_o(s) = H(s)$, $v_{i+1}(s) = \frac{s - si}{vi(s) - ai}$

And in the Matlab written as:

C = matsudaCFE (w, gain)

While Matsuda method the expansion given as:

$$(15) \quad H(s) = ao + \frac{s-so}{a1+} \frac{s-s1}{a2+} \frac{s-s2}{a3+}$$

$$A_i = v_i(s_i), \quad v_o(s) = H(s), \quad v_{i+1}(s) = \frac{s-si}{vi(s)-ai} \qquad (16)$$

And in the Matlab written as:

C = matsudaCFE (w, gain)

In additional other approximation mentioned in [7].The frequency response for both real transfer function for $S^{0.9}$ and the approximated method are given in fig (6) ,(7),(8).

```
clc
clear
a=1;
na=0;
b=1;
nb=0.9;
G=fotf(a,na,b,nb)
k=1;
a=10/9;
w=1;
N=5;
G2 = carlson(k, a, w, N);
f=minreal(G2)
bode(G)
bode(f)
```
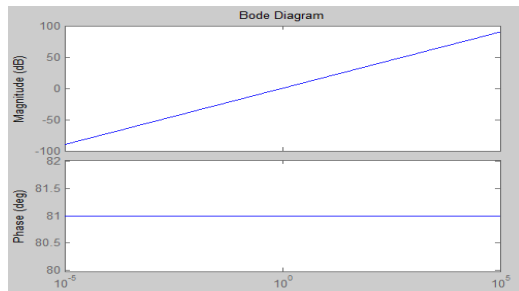
Figure (6): M.file Screen In Mat lab

Figure (7): Bode Diagram Of Real Transfer Function (G)
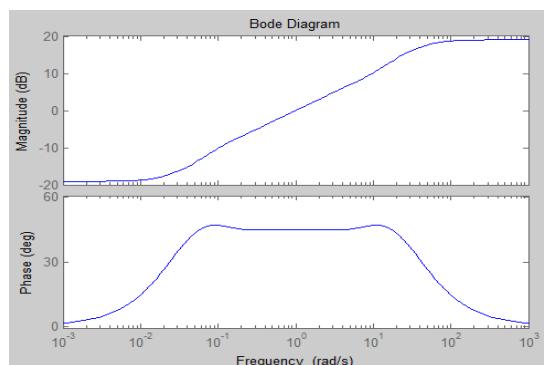
$$G = S^{0.9}$$

Figure (8): Bode Diagram Of Approximated Transfer Function(f)

It is clear the Bod diagram of approximated transfer function by Carlosn method look like at band width
$(10^{-2}\_10^2)$ rad/sec

IV.   ANALOG REALIZATION

After get the transfer functions by Carlson method it synthesized or converted to analog circuit that realizes it. In order to be applied in the required control circuit .There are two way for conversion either by use Foster form(first or second) or use Ladder form (Cauer second or first) as Structure implementation the transfer function.

4.1. Ladder form

The simplest procedure of fractance synthesis is based on the repeated division process of term Corresponding decomposition into the continued fraction in the most fundamental form[8][9] is:-

$$Y(s) = C_1 s + \cfrac{1}{R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_2 + \cfrac{1}{C_3 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \cfrac{1}{R_4}}}}}}}$$

Where Ci and Ri should be rounded to obtain real valued components.
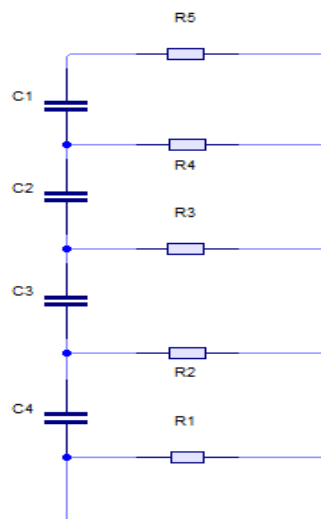

Figure (9): Ladder Circuit(second cure) Structure Of Fourth-order Admittance Approximant

The method has been applied for $s^{0.9}$ by (Matlab Rb2012).The synthesis to the analog circuit which correspond to the transfer function by use either continue fraction method(Caure) or partial fraction foster method .utilize Fomcon tool in (Matlab12) as in fig(9-a).

```
1 -     b1=[9 84 126 36 1];
2 -     a1=[1 36 126 84 9];
3 -     [q, expr]=polycfe(b1,a1)
```

Figure (9-a): M.file Window First Cure Form

$$9 + \cfrac{1}{-0.00416s + \cfrac{1}{-7.5472 + \cfrac{1}{-0.39901s + \cfrac{1}{-0.9091 + \cfrac{1}{-5.196s + \cfrac{1}{-0.28783 + \cfrac{1}{-44.9847s + \cfrac{1}{-0.14478}}}}}}}}$$
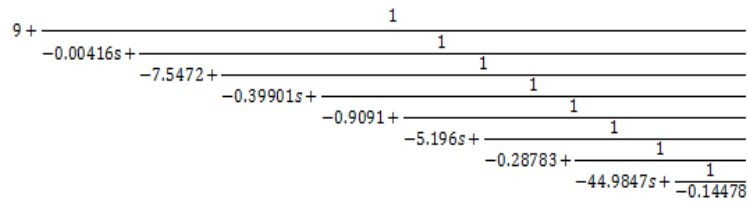
Figure (9-b): Result First Cauer form

Since this negative impedance, so use 2sd Coure method when the diviton preformed for lower order to avoid the negative term.

```
1 -    b1=[1 36 126 84 9];
2 -    a1=[9 84 126 36 1];
3 -    [q, expr]=polycfe(b1,a1)
```
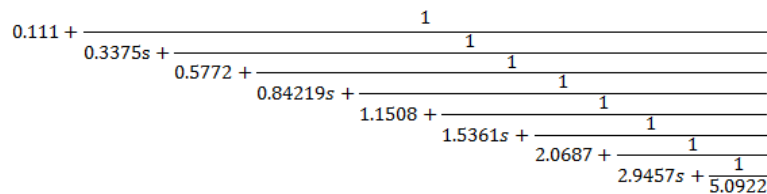
Figure (9-c): M.file Second Cure Form.

$$0.111 + \cfrac{1}{0.3375s + \cfrac{1}{0.5772 + \cfrac{1}{0.84219s + \cfrac{1}{1.1508 + \cfrac{1}{1.5361s + \cfrac{1}{2.0687 + \cfrac{1}{2.9457s + \cfrac{1}{5.0922}}}}}}}}$$

Figure (9-d): Result For Second Cure Form

Table (1) Shows Resistance And Capacitance Value

For Second Cure Method

R1	1/0.1111=9Ω	C1	1/0.3375=2.62 F
R2	1/0.5772=1.732 Ω	C2	1/0.84219=1.1873F
R3	1/1.1508=0.868 Ω	C3	1/1.5361=0.6509F
R4	1/2.0687=0.4833 Ω	C4	1/2.9457=0.3394F
R5	1/5.0922=0.1963 Ω		

When choosing the parameter Kd as unity and the

Kd given [9] as:

Kd =Zf /Zi

The equivalent impedance Zf for the fractance circuit in fig (9) ,found by Matlab command :-

```
clc
clear
R1=9; R2= 1.732; R3=0.868; R4=0.4833; R5=0.1963;
w=2*3.14*100;
s=j*w;
c1=1/(2.62*s);c2=1/(1.1873*s);c3=1/(0.6509*s); c4=1/(0.3394*s);
z=R5+4;z1=z*R4/(z+R4);z2=z1+c3;z3=z2*R3/(z2+R3);z4=z3+c2;z5=z4*R2/(z4+R2);
z6=z5+c1;
Zf=z6*R1/(z6+R1)
Zf= 0.2411 - 0.0023i
```

R_{in}=0.2411Ω
C_{in}=0.0023 F

4.2 Foster form

Which has two type first and second [10]. And the second has been applied. This method depends on the partial fraction expansion. Fractional expansion of the T.F has been utilized Mat lab program as given fig (10), (11) and has circuit structure as fig (14).become the result expansion in Eq (17)

$$0.111 + \frac{1.8997}{s + 7.5486} + \frac{0.5378}{s + 1.4203} + \frac{0.2963}{s + 0.3333} + \frac{0.2291}{s + 0.0311} \qquad (17)$$

Can implement as parallel combination of RC circuit correspond to each term of Eq (17).when the impedances as:

$$\frac{1/c}{s+1/Rc} \qquad (18)$$
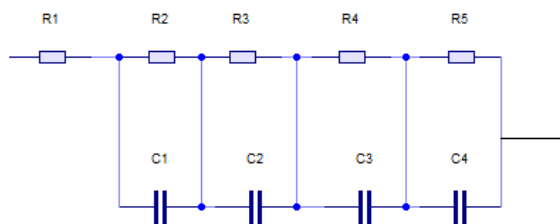
Correspond

Figure (10): Foster Form Circuit

The syntheses that correspond the term eq (17) and the circuit element given in table 2

```
1 -    clc
2 -    a=[9 84 126 36 1];
3 -    b=[1 36 126 84 9];
4 -    tf(a,b)
5 -    [r,p,k]=residue(b,a)
```

Figure (10-a): M.file Window In Matlab12 Second Foster Form

The result of residue is:

```
r =

     1.8997
     0.5378
     0.2963
     0.2291


p =

    -7.5486
    -1.4203
    -0.3333
    -0.0311


k =

     0.1111
```

Figure (10-b): Command Window In Mat lab

Table (2) Shows Resistance And Capacitance Value
For First Foster Method

R1	0.1111	C1	1.8997
R2	3.9728	C2	0.5378
R3	2.6409	C3	0.2963
R4	1.1248	C4	0.2291
R5	0.1357		

When choosing the parameter

Kd as unity and the

Kd given [9] as:

Kd =Zf /Zi

The equivalent impedance Zf for the fractance circuit in fig (11) ,found by Matlab command :-

clc

clear

R1=0.1111; R2= 3.9728; R3=2.6409; R4=1.1248; R5=0.1357;

w=2*3.14*100;

s=j*w;

c1=1/(1.8997*s);c2=1/(0.5378*s);c3=1/(0.2963*s); c4=1/(0.2291*s);

z=(c1*R2)/(c1+R2);z1=(c2*R3)/(c2+R3);z2=(c3*R4)/(c3+R4);

z3=(c4*R5)/(c4*R5);

Zf=R1+z3+z2+z1+z

the result appear

Zf=1.1111-0.0092i

$R_{in=}$1.1111Ω

C_{in}=0.0092 F

Notes:

When first Caure method applied some of value are appears negative, to resolve this problem will use operation amplifier which called Negative impedance converter (NIC) given in fig (13) Which both R have the same value and Z represent negative value[2]
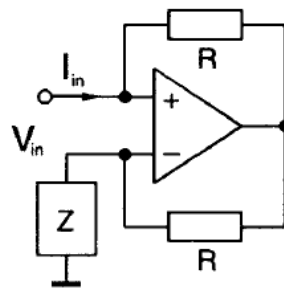

Figure (11): Negative Impedance Converter.

V.   SIMULATION IN CIRCUIT WIZARD

The synthesized circuit for the fractional order derivative 0.9 that contain the fractance which contain RC circuit as the foster method has been simulated using wizard circuit shown in Fig (14),(15).the square wave signal applied to the input of the circuit as given in Fig(14-a),(15-4) which is realization of fractance derivative circuit ,and the output appear as triangle signal shown Fig(14-a),(15-a). And when applying sin wave signal in input the output signal given as in fig(16) and compared with input signal we observe a phase shift and attenuation in amplitude, this phase shift value correspond to (α) and same as for attenuation.
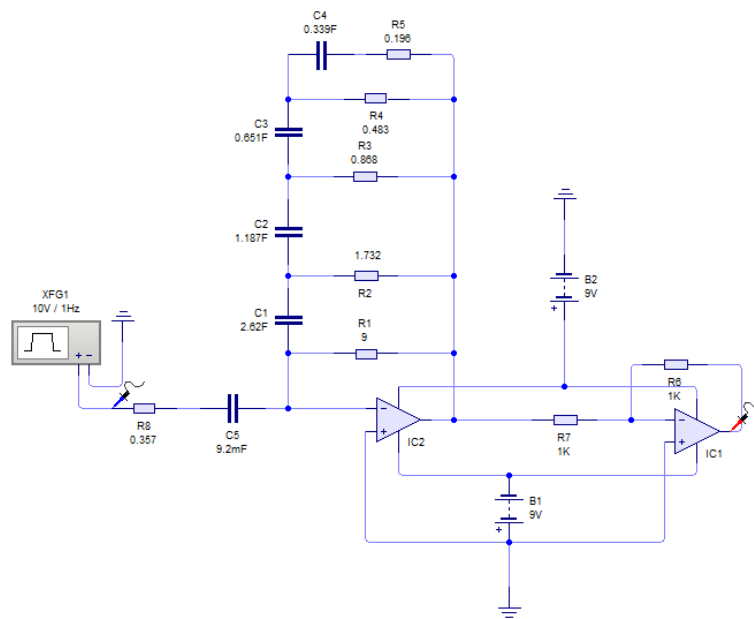
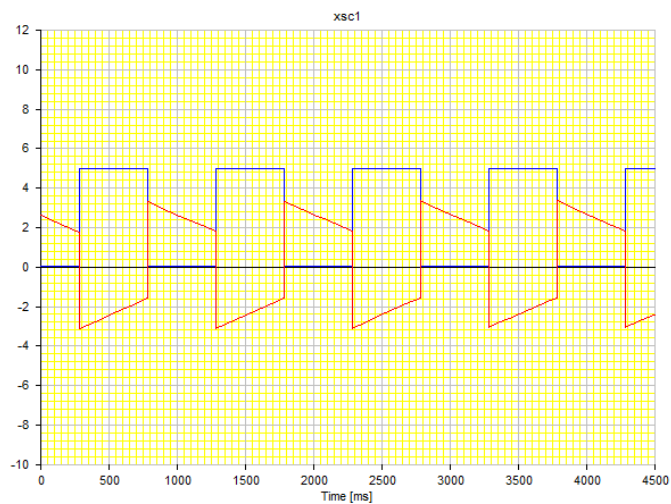**Figure (12):** Analog Realization For Fractance (Ladder Form) Derivative Type

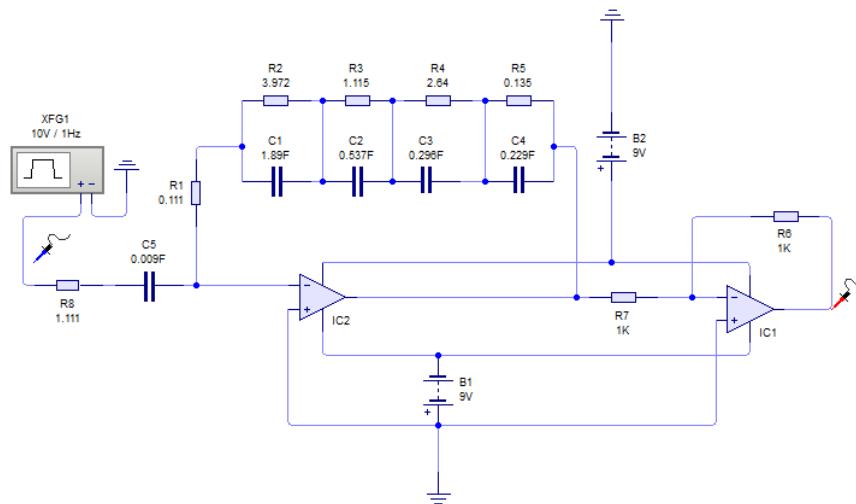**Figure (12-a):** Blue Line Input Signal ,Red Line Output Signal

Figure (13): Analog Realization For Fractance (Foster Form) Derivative Type
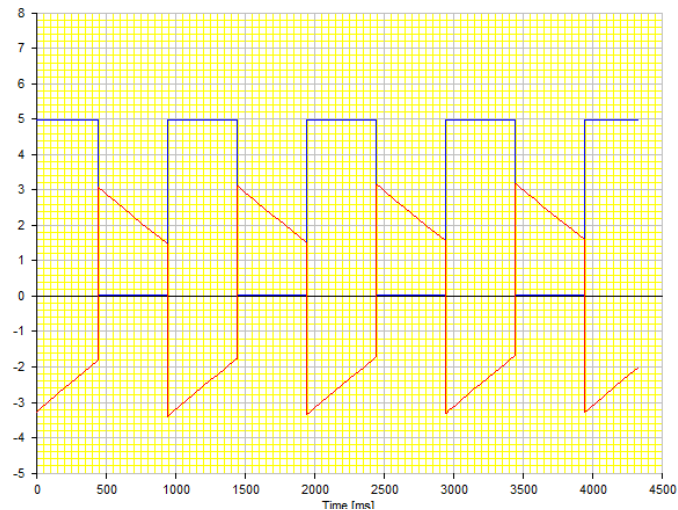
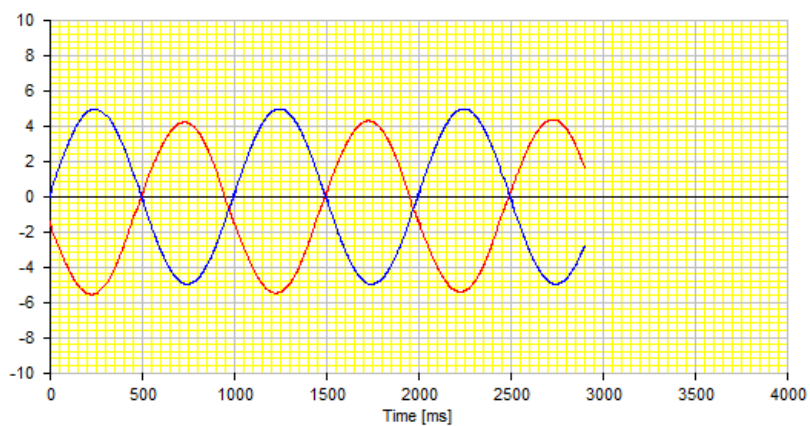

Figure (13-a): Blue Line Input Signal ,Red Line Output Signal

Figure (14): Red Line Input Signal, Blue Line Output Signal

VI.   CONCLUSION

The fractional order differentiation raised to non integer differential ($D^α$) or in Laplace operator ($S^α$) .The benefit of FOD is it has constant phase($α$ $π/2$) for wide band and the magnitude slop for it ($α$20db/dec) while the general integer derivative is(20db/dec) in the Bode diagram. Continues fractional expansion and partial fraction expansion can be used to synthesis the fractance as passive RC circuit in electrical circuit which has the form of ladder, tree and also transmission line. So it can be used in FPI^{λ}D^α controller. The synthesized circuit realized by electronic circuit utilized Wizard Circuit packages. A comparison between real integer derivative and 0.9 fraction derivative has well approximated in the chose band width taking into account magnitude and phase.

VII.   FUTURE WORK

  Some of the proposed future work given below:
1- Use other fractances like the tree structure, and transmission lines structure.
2- Use other methods for approximation, such as Crone and Matsuada.
3- Utilization of LabVIEW program instead of Matlab.
4- Use update electronic simulation programs such as Proteus.
5- Use Operation Transconductance Amplifier (OTA) instead of the traditional operational amplifier.
6- FPAA can be used to achieve the fractional circuit.

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