ANALYTICAL SOLUTION FOR A GEOMETRICALLY AND MATERIAL NONLINEAR 2D TRUSS

Pedro F L Cunha
Department of Civil Engineering,
University of Brasilia, Campus Darcy Ribeiro,
Brasilia DF, CEP 70910-900, Brazil
pedro_flc@hotmail.com

ABSTRACT

Structural analyses are increasingly embracing complex approaches to better model real-world behaviours. Within this context, the current study explores nonlinear analyses specifically focused on planar trusses. The research underscores the distinctions and complexities arising from both geometric and material deviations from linear behaviours. This work presents an analytical solution for a material nonlinear 2D truss. The main purpose of this work is provide a robust and precise analytical solution, which encapsulate the intricate nodal behaviour throughout various equilibrium stages. Such solutions cater to a broad spectrum of materials, from those strictly adhering to linear elastic behaviours to others demonstrating more intricate non-linear characteristics. A comprehensive contrast is drawn between the singular effects of geometric non-linearity and the intertwined consequences of both geometric and material non-linearities. This comparative examination shows pronounced differences in the predicted structural responses. An integral component of the research was the comparison of our analytical outcomes with a sophisticated, finite element-based software. These comparisons validate our findings but also position this study as a pivotal reference for future structural analyses. The main conclusion is the significance of incorporating non-linear considerations, both geometric and material-based, in any rigorous structural analyses, ensuring outcomes that align more congruently with real-world observations.

KEYWORDS: *Geometric Non-linear Analysis, Material Non-linear Analysis, Analytical Solution, Planar Truss Structures, Material Behaviour.*

I. INTRODUCTION

Structural engineering is fundamentally concerned with understanding the behavior of structures under varying load conditions. Traditionally, this understanding has relied heavily on linear analyses, which, while computationally efficient, might not fully capture the multifaceted behavior exhibited in the real world. Such discrepancies are especially evident in structures prone to significant deformations or those made of materials with non-linear characteristics. Planar trusses, though seemingly straightforward, can manifest these non-linear complexities (Wood and Zienkiewicz [1]).

As structural engineering continually evolves, there's a growing need for a more intricate approach that genuinely addresses non-linear behaviors. This research veers from the traditional linear standpoint, zeroing in on non-linear analysis with a particular emphasis on planar trusses. We explore the intricacies resulting from both geometric and material deviations.

With the contemporary emphasis on the precision and reliability of structural predictions, it's vital to enhance our analytical methodologies to mirror real-world scenarios more accurately. This paper responds to that imperative. It seeks to offer a robust analytical solution, capturing the nuanced behavior throughout different equilibrium stages in truss structures. These stages span from simple linear elasticity to the more complex non-linear scenarios (Yang and Leu [2]).

Linear analysis, due to its computational simplicity, has been a cornerstone in academia for gauging displacements, rotations, and stresses within structures. However, the distinction lies in Geometric Non-linearity (GNL), where the equilibrium conditions shift based on the structure's altered configuration post partial load application. In turn, Material Non-linearity (MNL) is a phenomenon where materials deviate from linear behavior, such as the predictions of Hooke's Law, under certain load thresholds.

Notably, as Lacerda [3] underscores, various materials demonstrate non-linear behaviors ranging from plasticity to viscoelasticity and creep.

Junior [4] elaborates that non-linear structural analysis, encapsulating the equilibrium curve determination, can utilize diverse techniques from analytical solutions to iterative methodologies. Greco [5] postulates that the truss, given its inherent simplicity, is a prime candidate for studying non-linear dynamics. This research proposes an analytical resolution for both geometric and material non-linear analysis of a planar truss, positioning this solution as a benchmark for the comparison of numerical approaches.

Subsequent sections will immerse readers in the intricate world of non-linear behaviors, seamlessly blending theoretical constructs with numerical comparison, emphasizing the pivotal role of acknowledging both geometric and material non-linear facets in contemporary structural engineering.

The following sections will immerse readers in the complex realm of non-linear behaviors, seamlessly integrating theoretical concepts with numerical comparisons. This will underscore the crucial importance of considering both geometric and material non-linear aspects in structural engineering.

To achieve this, the methodology will address two scenarios. The first scenario involves considering geometric non-linearity alongside linear elastic. The second scenario will present a planar truss with geometric non-linearity and one of its members exhibiting non-linear material behavior. In both cases, equations will be developed to calculate displacements and forces in the members for various load levels. Finally, in the results section, horizontal and vertical displacements, as well as member forces, will be computed for both scenarios, with a continuous comparison to numerical results obtained using the finite element Amaru software (https://github.com/NumSoftware/Amaru.jl/tree/main).

II. RELATED WORKS

Despite being a very complex topic, the nonlinear analysis of structures attracts various researchers who seek to develop work both to find analytical solutions and numerical responses to various engineering problems. Following are some works developed in the last six years that address nonlinear analysis in flat structures.

Liu and Lv [6] introduced an equivalent continuum multiscale approach for geometrically nonlinear analysis of lattice truss structures. It combines the multiscale finite element method and co-rotational approach. The lattice truss unit cell is approximated as a continuum coarse element, and its tangent stiffness matrix is derived. This approach captures multiple critical points in the equilibrium path, and microscopic information can be obtained efficiently. Numerical examples examine unit cell layout, size, and mesh sensitivity, confirming the method's validity and efficiency.

The paper developed by Rezaiee-Pajand and Naserian [7] introduces an iterative approach for nonlinear analysis using triangular shapes derived from load-displacement curves. These shapes represent objective functions to minimize, yielding two constraint equations for nonlinear solving. The method is applied to geometric nonlinear analyses of shells, frames, and trusses, with a comparison to the cylindrical arc-length method to demonstrate its effectiveness.

Habib and Bidmeshki [8] developed a dual approach for geometrically nonlinear finite element analysis of plane truss structures. It employs the Total Lagrangian formulation to account for geometric nonlinearity and introduces an objective function to minimize displacement-type constraints. The method traces the entire equilibrium path, eliminating errors caused by linearization, and can predict pre- and post-buckling behavior and multiple limit points with snap-back. Numerical results demonstrate its accuracy and efficiency, validated against theoretical solutions and existing methods in the literature.

The paper developed by Falope et al. [9] examined the equilibrium and stability of a von Mises truss under vertical load, using theoretical, numerical, and experimental approaches. The truss is made of rubber, allowing for large deformations. The study uses a fully nonlinear finite elasticity model with the Mooney-Rivlin law to characterize the rubber's behavior, identified via genetic algorithms. Experimental observations confirm snap-through behavior, validating the nonlinear approach's accuracy in predicting snap-through and Eulerian buckling, contrasting with linear elasticity method

Perônica et al [10] focused on developing a computational code to analyze and compare the mechanical behavior of trusses with hyperelastic materials, considering both physical and geometric nonlinearities. Various hyperelastic models are considered, and the code's comparison is achieved by comparing it with analytical, numerical, and experimental results from scientific papers.

Fonseca and Gonçalves [11] studied investigates the nonlinear behavior, bifurcations, and instabilities of a hyperelastic von Mises truss, aiming to achieve multistable behavior. Unlike previous research focused on linear elastic materials, this work considers fully nonlinear elasticity with the incompressible Mooney-Rivlin constitutive law. The Newton-Raphson method and continuation techniques are used to solve the nonlinear equations, revealing multiple equilibrium paths and stability points. Geometric and material parameters, as well as load and imperfections, are analyzed, resulting in coexisting stable and unstable solutions. Analytical expressions for snap-through and pitchfork bifurcation loads are derived. The findings have implications for engineering applications requiring multistability and large deformations.

III. METHODOLOGY

The methodology used in this paper is based on the work of Greco [5]. We present the nonlinear analyses of a 2-bar truss loaded at node 2 as shown in Fig. 1.

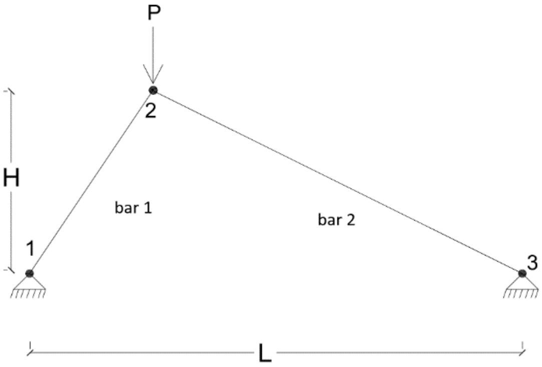

Figure 1. Undeformed truss of the Von Mises type.

Initially, before the application of load P, the truss exhibits an undeformed structure with angles α_0 and β_0. After the concentrated load is applied the bars deform as shown in Figure 2.

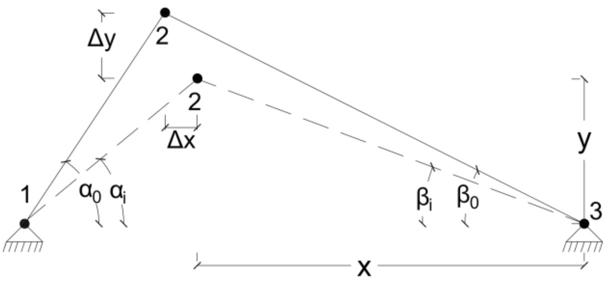

Figure 2. Deformed truss of the Von Mises type.

Utilizing the equilibrium equations at the central node and considering the sum of horizontal forces to be zero, we obtain:

$$F_1 = \frac{cos\beta_i}{cos\alpha_i} F_2 \tag{1}$$

Applying the equilibrium equation in the vertical dxxxx yields:

$$F_1 sen\alpha_i + F_2 sen\beta_i = P \qquad (2)$$

Substituting Eq. (1) into (2) we find the axial force in the second bar:

$$F_2 = \frac{P}{cos\beta_i tan\alpha_i + sen\beta_i} \qquad (3)$$

By integrating the kinematics of the problem, we can derive two equations that correlate the displacements of bars 1 and 2, respectively:

$$\delta_1 = \frac{(L-x)}{cos\alpha_i} - \frac{a}{cos\alpha_0} \qquad (4)$$

$$\delta_2 = \frac{x}{cos\beta_i} - \frac{b}{cos\beta_0} \qquad (5)$$

2.1. Analytical Solution Considering Geometric Non-linearity and Linear Elastic Material

Initially, the material is characterized as linear elastic. By applying Hooke's Law to bar 1 within the deformed structure, we have:

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} \qquad (6)$$

Thus,

$$F_1 = \delta_1 E_1 A_1 \frac{cos\alpha_0}{a} \qquad (7)$$

Substituting Eq. (3) into (1):

$$F_1 = \frac{cos\beta_i}{cos\alpha_i}\left(\frac{P}{cos\beta_i tan\alpha_i + sen\beta_i}\right) \qquad (8)$$

Then substituting Eq. (4) and Eq. (8) into Eq. (7) we get:

$$\frac{cos\beta_i}{cos\alpha_i}\left(\frac{P}{cos\beta_i tan\alpha_i + sen\beta_i}\right) = E_1 A_1 \left[\frac{cos\alpha_0 (L-x)}{a(cos\alpha_i)} - 1\right] \qquad (9)$$

Similarly, using Hooke's Law again for bar 2 we have:

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} \qquad (10)$$

$$F_2 = \delta_2 E_2 A_2 \frac{cos\beta_0}{b} \qquad (11)$$

Thus, substituting Eqs. (3) and (5) into Eq. (11) we arrive to:

$$\frac{P}{cos\beta_i tan\alpha_i + sen\beta_i} = E_2 A_2 \left(\frac{x(\,cos\beta_0)}{b(cos\beta_i)} - 1\right) \quad (12)$$

Noted that the left-hand term of Eq. (12) is incorporated into Eq. (9). This allows us to find an expression independent of external loading, as given by Eq. (13):

$$\frac{cos\beta_i}{cos\alpha_i}\left[\left(\frac{x(\,cos\beta_0)}{b(cos\beta_i)} - 1\right)\right] = \frac{E_1 A_1}{E_2 A_2}\left[\frac{cos\alpha_0(L-x)}{a(cos\alpha_i)} - 1\right] \quad (13)$$

Substituting Eq. (5) into Eq. (11):

$$F_2 = \left[\frac{x(cos\beta_0)}{b(cos\beta_i)} - 1\right] E_2 A_2 \quad (14)$$

Using Eqs. (15) and (16), derived from trigonometric relationships in Figure 2, we can determine the values for x and y. As:

$$x = \frac{L\tan\alpha_i}{\tan\beta_i + \tan\alpha_i} \qquad (15)$$

$$y = \frac{L\tan\alpha_i \tan\beta_i}{\tan\beta_i + \tan\alpha_i} \qquad (16)$$

Finally, substituting Eq. (15) into Eq. (13) we obtain an analytical expression for β_i as:

$$\left[\frac{cos\beta_i}{cos\alpha_i}\left(\frac{\frac{L\tan\alpha_i}{\tan\beta_i + \tan\alpha_i} cos\beta_0}{b(cos\beta_i)} - 1\right)\right] = \frac{E_1 A_1}{E_2 A_2}\left[\frac{cos\alpha_0\left(L - \frac{L\tan\alpha_i}{\tan\beta_i + \tan\alpha_i}\right)}{a(cos\alpha_i)} - 1\right] \qquad (17)$$

The input parameters for this analytical solution are: $L, a, b, E_1, E_2, A_1, A_2$ and α_i. By substituting these values into Eq. (17), the value of β_i can be found.

2.2. Analytical Solution Considering Geometric and Material Non-linearity.

For this study, bar 1 is considered from a non-linear material, while bar 2 exhibits linear elastic behaviour. The constitutive law for the non-liner material will be represented by the following hypothetical equation:

$$\sigma_1 = E_1\left(\varepsilon_1 - 3{\varepsilon_1}^2\right) \quad (18)$$

Figure 3 illustrates the stress-strain curve corresponding to Eq. (18):

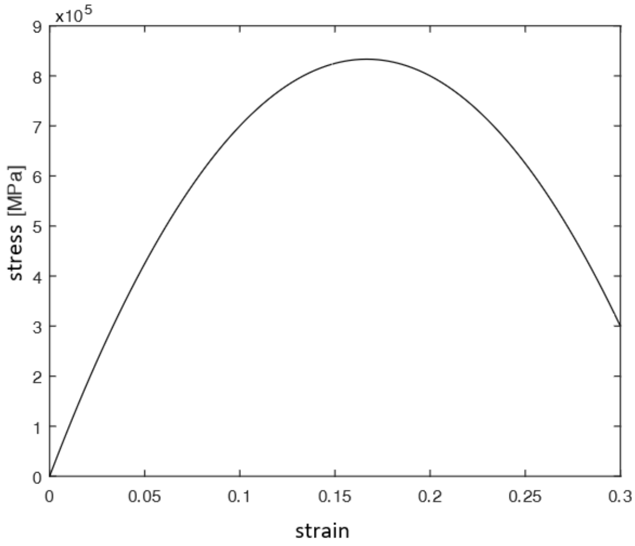

Figure 3. Stress versus strain curve for the non-linear material.

Applying Eq. (18) to bar 1 we have:

$$F_1 = A_1 E_1 \left(\varepsilon_1 - 3{\varepsilon_1}^2\right) \quad (19)$$

Where:

$$\varepsilon_1 = \frac{L \sin\beta_i cos\alpha_0}{a[sin(\alpha_i + \beta_i)]} - 1 \quad (20)$$

By substituting Eqs. (14) and (19) into Eq. (1):

$$A_1 E_1\left(\varepsilon_1 - 3{\varepsilon_1}^2\right) = \frac{cos\beta_i}{cos\alpha_i} A_2 E_2 \left[\frac{x(cos\beta_0)}{b(cos\beta_i)} - 1\right] \quad (21)$$

Finally, incorporating Eqs. (15) and (20) into Eq. (21) we set the following analytical expression:

$$\left[\left(\frac{L \sin\beta_i cos\alpha_0}{a[sin(\alpha_i+\beta_i)]} - 1\right) - 3\left(\frac{L \sin\beta_i cos\alpha_0}{a[sin(\alpha_i+\beta_i)]} - 1\right)^2\right] = \frac{cos\beta_i}{cos\alpha_i}\frac{A_2E_2}{A_1E_1}\left[\frac{\frac{L\tan\alpha_i}{\tan\beta_i+\tan\alpha_i}cos\beta_0}{b(cos\beta_i)} - 1\right] \quad (22)$$

The input parameters for calculating the angle β_i are the same as those used in the analysis for the linear elastic material.

IV. Results

The geometric and material specifications for the truss discussed earlier are detailed in Table 1.

Table 1. Geometric and material properties.

A1 (m²)	A2 (m²)	E1 (GPa)	E2 (GPa)
1,00E-03	1,00E-03	10	10

Using equation (17), one can determine values for β_i , and subsequently, through Eqs. (15) and (16), ascertain the values of x and y. By subtracting these from the length L and height H, respectively, we obtain Δx and Δy increments. Eq. (11) then provides the force in bar 2. Later, the force in bar 1 and the applied force P are deduced using Eqs. (1) and (2). Figure 4 displays the obtained results for the horizontal displacement of node 2. These results were compared with the ones obtained from the Amaru software, a finite element solution developed using the high-performance programming language Julia. It is observed that in terms of horizontal displacement, the FEM solution proved to be more rigid compared to the analytical response. This can be explained using displacement control as a tool for the numerical solution of the nonlinear system.

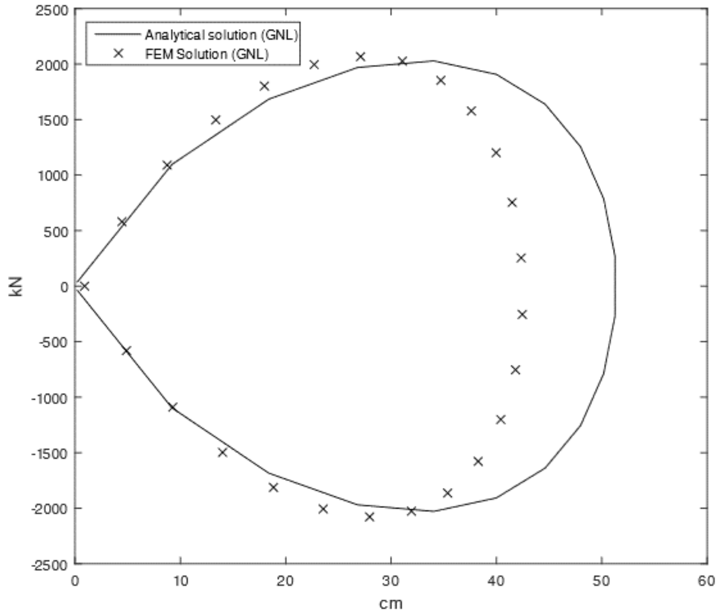

Figure 4. Comparison of the results obtained for the horizontal displacement of node 2 for the GNL.

The same finite element program was used to validate the vertical displacement of node 2, as shown in Figure 5.

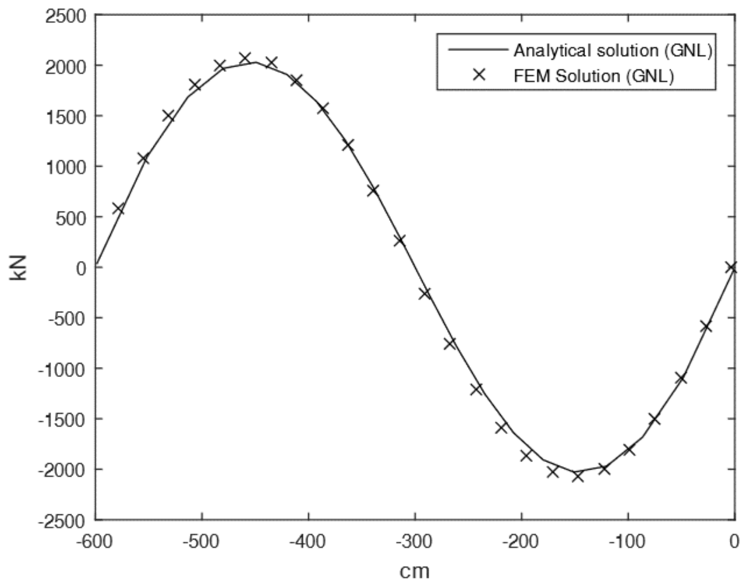

Figure 5. Comparison of the results obtained for the vertical displacement of node 2 for GNL and MNL.

Similarly, by performing the same calculation procedure, and considering bar 1 as non-linear, the following displacements were obtained:

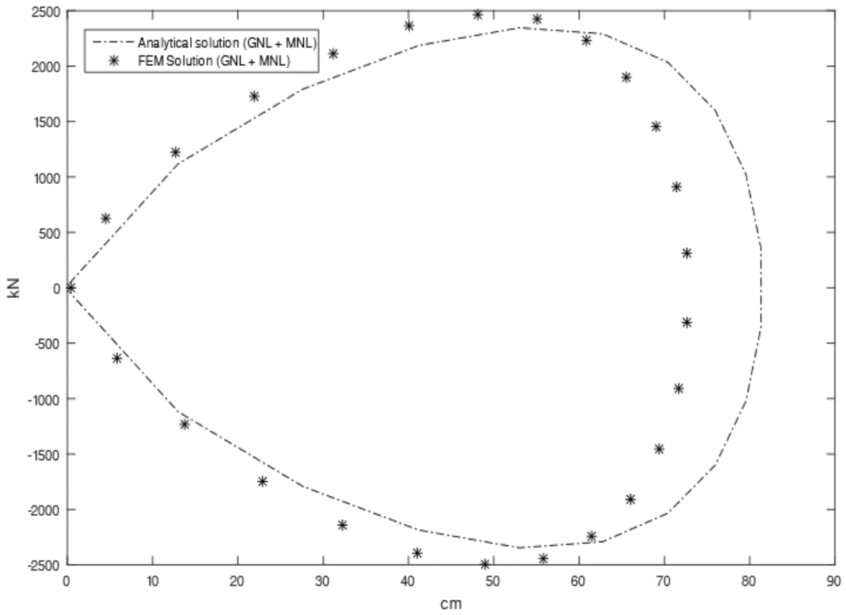

Figure 6. Comparison of the results obtained for the horizontal displacement of node 2 for GNL and MNL.

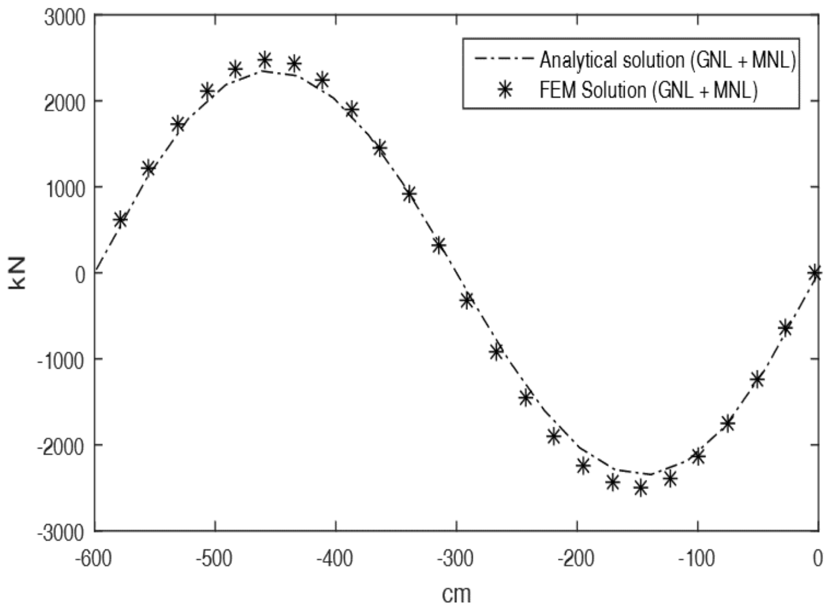

Figure 7. Comparison of the results obtained for the vertical displacement of node 2 for GNL and MNL.

Figures 8 and 9, the results from the analysis considering only the geometric non-linearity are compared with those considering both GNL and MNL:

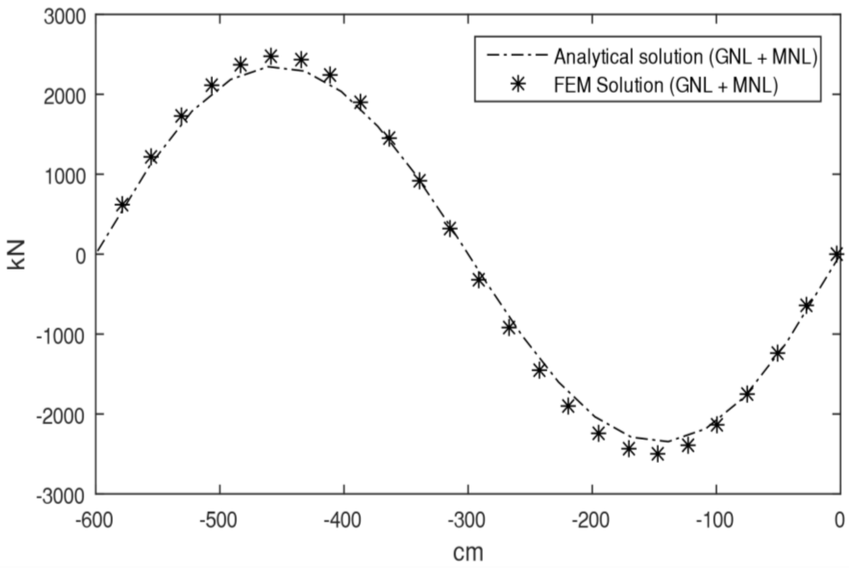

Figure 8. Comparison between GNL solution and GNL + MNL for the horizontal displacement of node 2.

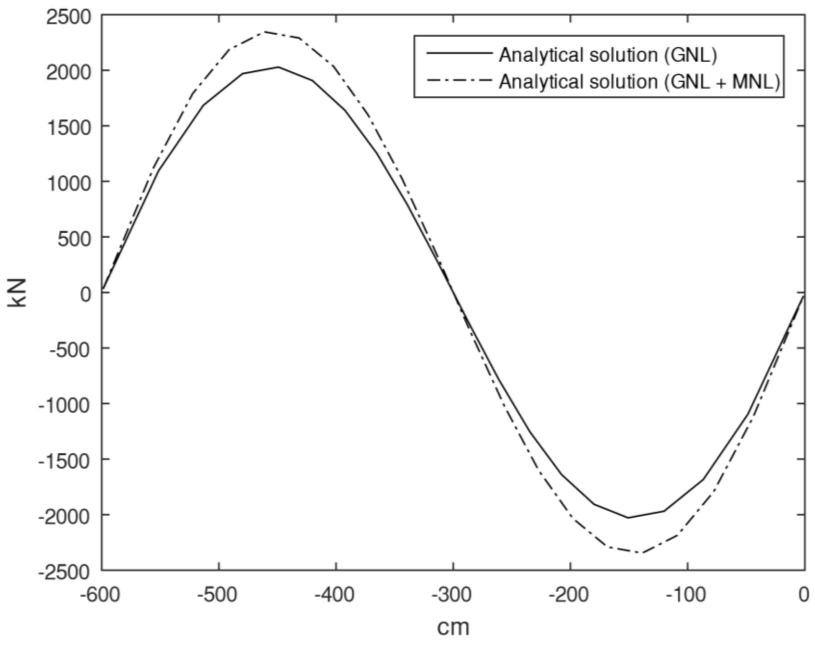

Figure 9. Comparison between GNL solution and GNL+ MNL for the horizontal displacement of node 2.

In Figure 10, the equilibrium path for node 2 is examined under both scenarios:

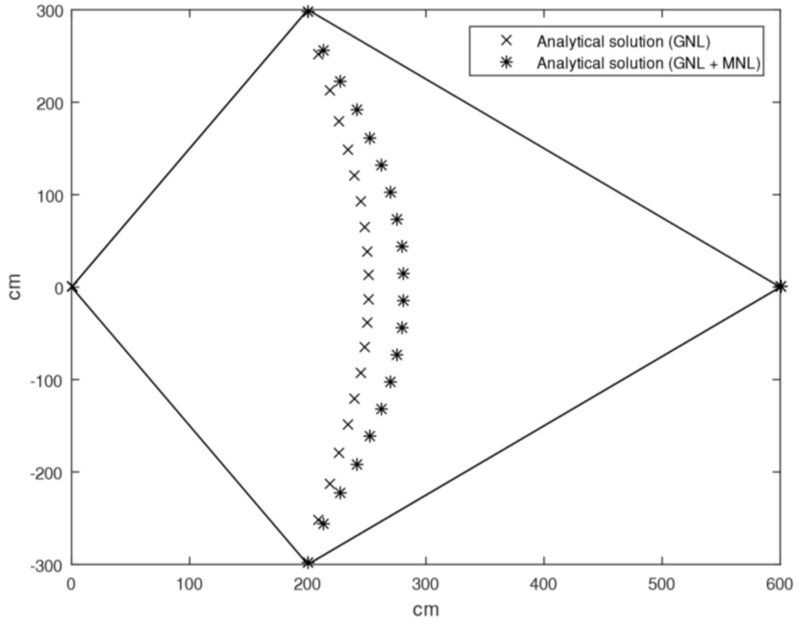

Figure 10. Comparison between the GNL analysis and GNL+MNL analysis.

Through Figures 11 and 12, we compare the intensities of the axial forces in the bars over the horizontal and vertical displacement of the node, respectively:

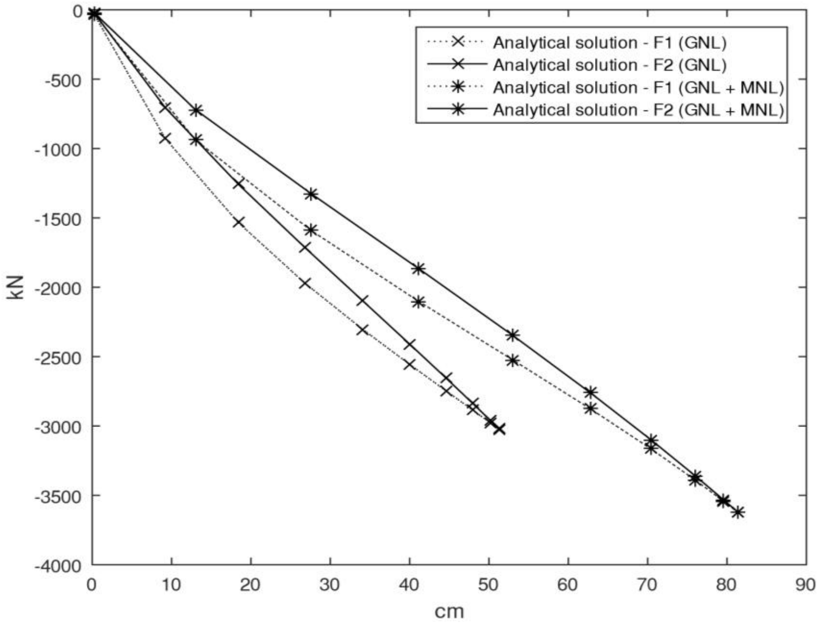

Figure 11. Comparison between the GNL solution and GNL+MNL for the horizontal displacement of node 2 and the forces in the bars.

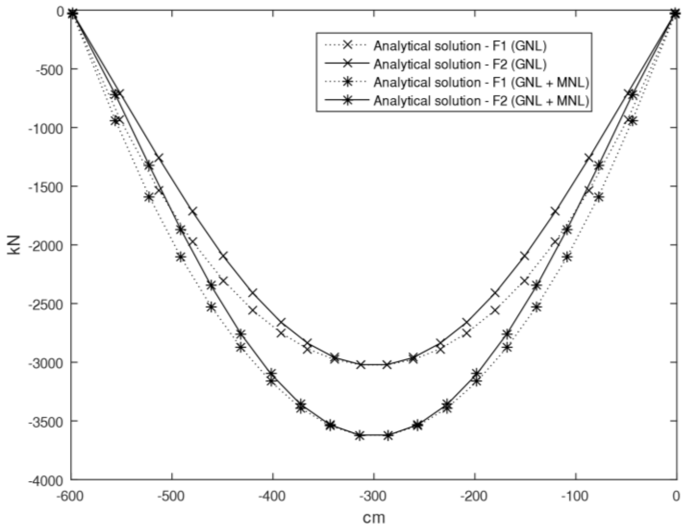

Figure 12. Comparison between the GNL solution and GNL+MNL for the vertical displacement of node 2 and the forces in the bars.

V. CONCLUSIONS

This study aimed to provide analytical solutions for planar trusses under both geometrically and materially non-linear analyses. Despite the seemingly simplistic nature of the truss often considered as a foundational model in academic research, the results underscore the depth and complexity of non-linear behavior, both from a geometric and material standpoint.

The inclusion of both geometric and material non-linearity in the analyses, as demonstrated, can significantly alter the predicted values for nodal displacements and internal bar forces. This fact emphasizes the importance of considering these non-linearities, especially in situations where the structure may be subject to large displacements or when the material does not follow a linear relationship, such as Hooke's law.

The contrast shown between the results from the analysis considering only geometric non-linearity and those incorporating both non-linearities (GNL and MNL) is enlightening. It reveals that by ignoring any of these non-linearities in an analysis, one can significantly under or overestimate the structure's responses.

The comparison of the proposed analytical solutions with a finite element-based software (Amaru) highlights the accuracy and applicability of the formulations presented. This kind of comparison is vital to establish trust in the analytical solutions, especially when they are proposed as a benchmarking tool for numerical methods.

Lastly, it's crucial to note that while this study focused on a specific planar truss, the implications are much broader. The deep understanding and the ability to adequately model non-linear behaviors are essential in many domains of structural and civil engineering, from building and bridge design to the analysis of aerospace components.

The work presented not only offers a robust methodology for analyzing non-linear trusses but also underscores the ongoing need for research and development in analytical and numerical methods that can accurately capture the real behavior of structures under varied loadings.

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AUTHORS

Pedro Filipe de Luna Cunha

Ph.D. candidate in Structures at the University of Brasília (UnB). Master’s degree in civil engineering with a concentration in Structures from the Federal University of Pernambuco (UFPE). Specialist in Structural Design, Execution, and Control of Structures and Foundations from IPOG. Completed a bachelor’s degree in civil engineering at the Federal University of Campina Grande (UFCG).