FRACTAL CHARACTERIZATION OF EVOLVING TRAJECTORIES OF DUFFING OSCILLATOR

Salau, T. A.O.[1] and Ajide, O.O.[2]
[1, 2] Department of Mechanical Engineering, University of Ibadan, Nigeria.

ABSTRACT

This study utilised fractal disk dimension characterization to investigate the time evolution of the Poincare sections of a harmonically excited Duffing oscillator. Multiple trajectories of the Duffing oscillator were solved simultaneously using Runge-Kutta constant step algorithms from set of randomly selected very close initial conditions for three different cases. These initial conditions were from a very small phase space that approximates geometrically a line. The attractor highest estimated fractal disk dimension was first recorded at the end of 15, 22, and 5 excitation periods for Case-1, Case-2 and Case-3 respectively. The corresponding scatter phase plots for Case-1 and Case-2 agreed qualitatively with stroboscopic-ally obtained Poincare sections found in the literature. The study thus established sensitivity of Duffing to initial conditions when driven by different combination of damping coefficient, excitation amplitude and frequency. It however showed a faster, accurate and reliable alternative computational method for generating its Poincare sections.

KEYWORDS: *Duffing oscillator, Fractal, Poincare sections, Trajectories, Disk dimension, Runge-Kutta and phase space*

I. INTRODUCTION

Duffing oscillator can be described as an example of a periodically forced oscillator with a nonlinear elasticity [14].This can be considered as chaotic system since it is characterized by nonlinearity and sensitivity to initial conditions. Available literature shows that Duffing oscillator has been highly studied and this is due to its wide modelling applications in various fields of dynamics. The dynamics of duffing oscillator has been studied using various tools. [9] investigated the dynamical behaviour of a duffing oscillator using bifurcation diagrams .The results of the study revealed that while bifurcation diagram is a resourceful instrument for global view of the dynamics of duffing oscillator system over a range of control parameter, it also shows that its dynamics depend strongly on initial conditions. [11] Investigated the dynamic stabilization in the double-well Duffing oscillator using bifurcation diagrams. The research paper identified an interesting behaviour in the dynamic stabilization of the saddle fixed point. It was observed that when the driving amplitude is increased through a threshold value, the saddled fixed point. It was observed that when the driving amplitude is increased through a threshold value, the saddle fixed point becomes stabilized with the aid of a pitchfork bifurcation. The findings of the authors revealed that after the dynamic stabilization, the double-well Duffing oscillator behaves as the single –well Duffing oscillator. This is because the effect of the central potential barrier on the dynamics of the system becomes negligible.

A fractal generally refers to a rough or fragmented geometric shape which is capable of been divided into parts. Each part is an approximately reduced-size copy of the whole. This property is popularly referred to as 'self-similarity'. We can also describe fractal as geometric pattern that is repeated at ever smaller scales to produce irregular shapes and surfaces that cannot be represented by classical geometry. The complex nature of fractal is becoming to attract more researchers interest in the recent time. This is because it has become a major fundamental of nonlinear dynamics and theory of chaos.

Fractal structures and dynamical systems associated with phase plots are inseparable. The strong relationship between fractal structures and chaos theory will continue to remain the platform of success in nonlinear dynamics. Fractals are highly employed in computer modelling of irregular patterns and structures in nature. Though the theory of chaos and the concept of fractals evolved independently, they have been found to penetrate each other's front. The orbits of nonlinear dynamical system could be attracted or repelled to simple shape of nonlinear, near-circles or other shapes of Elucid[10].He furthered his explanation that,however,these are rare exceptions and the behaviour of most nonlinear dynamical systems tends to be more complictaed.The analysis of nonlinear dynamics fractal is useful for obtaining information about the future behaviour of complex systems [5] .The main reason for this is because they provide fundamental knowledge about the relation between these systems and uncertainty and indeterminism. [5] research paper focus on fractal structures in nonlinear dynamics. The work clearly describes the main types of fractal basin, their nature and the numerical and experimental techniques used to obtain them from both mathematical models and reap phenomena. [7] Research paper was on intermingled fractal arnold tongues. The paper presented a pattern of multiply interwoven Arnold tongues in the case of the single-well Duffing oscillator at low dissipation and weak forcing. It was observed that strips 2/2 Arnold tongues formed a truncated fractal and the tonguelike regions in between a filled by finely intermingled fractal like 1/1 and 3/3 Arnold tongues, which are fat fractals characterized by the uncertainty exponent alpha approximate to 0.7. The findings of authors showed that the truncated fractal Arnold tongues is present in the case of high dissipation as well, while the intermingled fractal pattern gradually disappears with increasing dissipation. [16] Research paper was on 1/3 pure sub-harmonic solution and fractal characteristic of transient process for Duffing's equation. The investigation was carried out using the methods of harmonic balance and numerical integration. The author introduced assumed solution and was able to find the domain of sub-harmonic frequencies. The asymptotical stability of the sub-harmonic resonances and the sensitivity of the amplitude responses to the variation of damping coefficient were examined. Then, the subatomic resonances were analyzed by using techniques from the general fractal theory. The analysis reveals that the sensitive dimensions of the system time-field responses show sensitivity to the conditions of changed initial perturbation ,changed damping coefficient or the amplitude of excitation. The author concluded that the sensitive dimension can clearly describe the characteristics of the transient process of the subharmonic resonances. According to [15] , the studies of the phenomenon of chaos synchronization are usually based upon the analysis of transversely stable invariant manifold that contains an invariant set of trajectories corresponding to synchronous motions. The authors developed a new approach that relies on the notions of topological synchronization and the dimension for Poincare recurrences. The paper showed that the dimension of Poincare recurrences may serve as an indicator for the onset of synchronized chaotic oscillations. The hallmark of [12] paper in 2007 was to examine the application of a simple feedback controller to eliminate the chaotic behaviour in a controlled extended Duffing system. The reason was to regulate the chaotic motion of an extended Duffing system around less complex attractors, such as equilibrium points and periodic orbits. The author proposed a feedback controller which consists of a high –pass filler and a saturator. This gives the opportunity of simple implementation and can be made on the basis of measured signals. The authors sufficiently demonstrated this feedback control strategy using numerical simulations. [8] Study was on characterization of non stationary chaotic systems. The authors noticed that significant work has not been done in the characterization of these systems. The paper stated that the natural way to characterize these systems is to generate and examine ensemble snapshots using a large number of trajectories, which are capable of revealing the underlying fractal properties of the system. The authors concluded that by defining the Lyapunov exponent and the fractal dimension based on a proper probability measure from the ensemble snapshots, the Kaplan-Yorke formula which is fundamental in nonlinear dynamics can be shown. This finding remains correct most of the time even for non- stationary dynamical systems.

Chaotic dynamical systems with phase space symmetries have been considered to exhibit riddle basins of attraction [1].This can be viewed as extreme fractal structures not minding how infinitesimal the uncertainty in the determination of an initial condition. The authors noticed that it is not possible to decrease the fraction of such points that will surely converge to a given attractor. The main aim of the authors' work was to investigate extreme fractal structures in chaotic mechanical systems. The authors investigated mechanical systems depicting riddle basins of attraction. That is, a particle under two-dimensional potential with friction and time-periodic forcing. The authors was able to verify this riddling by checking its mathematical requirements through computation of finite-time Lyapunov exponents as well as by scaling laws that explains the fine structure of basin filaments densely intertwined in phase space. A critical characterization of non-ideal oscillators in parameter space was carried out by [13].The authors investigated dynamical systems with non-ideal energy source. The chaotic dynamics of an impact oscillator and a Duffing oscillator with limited power supply were analyzed in two-dimensional parameter space by using the largest Lyapunov exponents identifying self-similar periodic sets, such as Arnold tongues and shrim-like structures. For the impact oscillator, the authors identified several coexistence of attractors showing a couple of them, with fractal basin boundaries. According to the paper, these kinds of basins structures introduce a certain degree of unpredictability on the final state. The simple interpretation of this is that the fractal basin boundary results in a severe obstruction to determine what attractor will be a fine state for a given initial condition with experimental error interval.

Fractal characterization of evolving trajectories of a dynamical system will no doubt be of immense help in diagnosing the dynamics of very important chaotic systems such as Duffing oscillator. Extensive literature search shows that disk dimension is yet to be significantly employed in fractal characterization of Duffing oscillator. The objective of this study is to investigate and characterize the time evolution of Poincare sections of a harmonically excited Duffing oscillator using fractal disk dimension.

This article is divided into four sections. Section 1 gives the study background and brief review of literature. Section 2 gives the detail of methodology employed in this research. Subsection 2.1 gives the equation of harmonically excited duffing oscillators that is employed in demonstrating fractal characterization of evolving trajectories. Subsection 2.1 gives explanation on the parameter details of all the studied cases. Different combinations of damping coefficient and excitation amplitude considered are clearly stated. The methodology is concluded in subsection 2.3 where explanation is given on how attractor is characterized. Section 3 gives detail results and discussion. The findings of this work are summarized in section 4 with relevant conclusions.

II. METHODOLOGY

2.1 *Duffing Oscillator*

The studied normalized governing equation for the dynamic behaviour of harmonically excited Duffing system is given by equation (1).

$$\ddot{x} + \gamma \dot{x} - \frac{x}{2}(1 - x^2) = P_o Sin(\omega t) \tag{1}$$

In equation (1) x, $\dot{x}$ and $\ddot{x}$ represents respectively displacement, velocity and acceleration of the Duffing oscillator about a set datum. The damping coefficient is γ. Amplitude strength of harmonic excitation, excitation frequency and time are respectively P_o, ω and t. [2], [3] and [6] proposed that combination of γ = 0.168, P_o = 0.21, and ω = 1 .0 or γ = 0.0168, P_o = 0.09 and ω = 1.0 parameters leads to chaotic behaviour of harmonically excited Duffing oscillator. This study investigated the evolution of 3000 trajectories that started very close to each other and over 25 excitation periods at a constant step ($\Delta t = \frac{Excitation\ period}{500}$) in Runge-Kutta fourth order algorithms. The resulting attractors (see [4]) at the end of each excitation period were characterized with fractal disk dimension estimate based on optimum disk count algorithms.

2.2 Parameter *details of studied cases*

Three different cases were studied using the details given in table 1 in conjunction with governing equation (1). Common parameters to all cases includes initial displacement range ($0.9 \le x \le 1.1$), Zero initial velocity ($\dot{x}$), excitation frequency (ω) and random number generating seed value of 9876.

Table 1: Combined Parameters for Cases

Cases	Damping coefficient (γ)	Excitation amplitude (P_o)
Case-1	0.1680	0.21
Case-2	0.0168	0.09
Case-3	0.0168	0.21

2.3 Attractor Characterization

The optimum disk count algorithm was used to characterize all the resulting attractors based on fifteen (15) different disk scales of examination and over five (5) independent trials.

III. Results and Discussion

The scatter phase plots of figures 1, 2 and 3 shows the comparative attractors resulting from the time evolution of trajectories of Duffing oscillator for the studied cases.

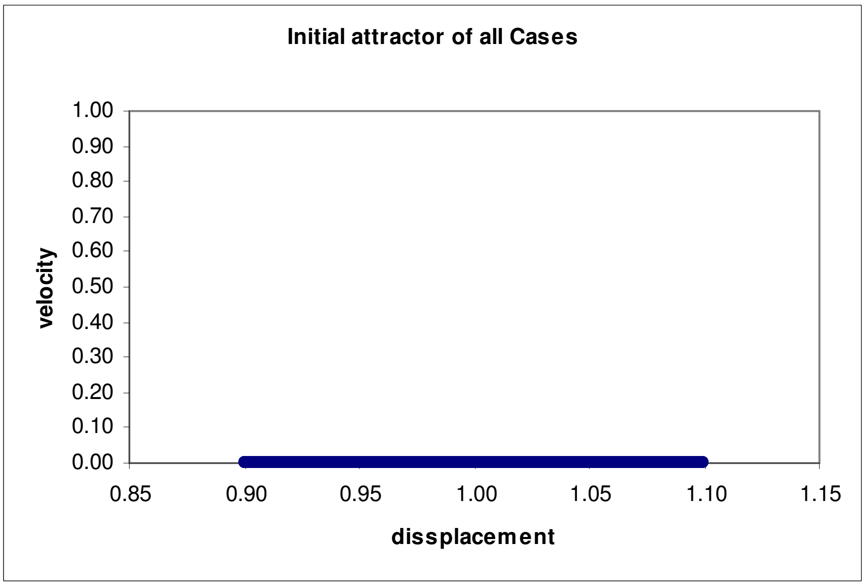


Figure 1: Attractor of all cases at zero excitation period.

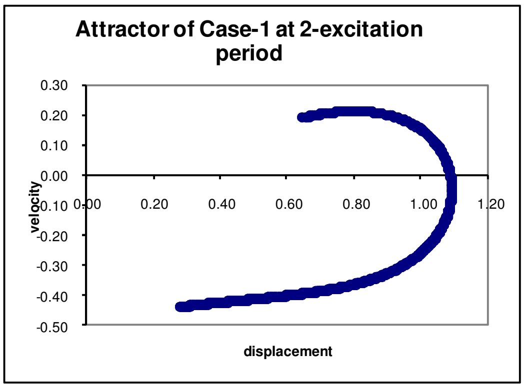


Fig. 2 (a)

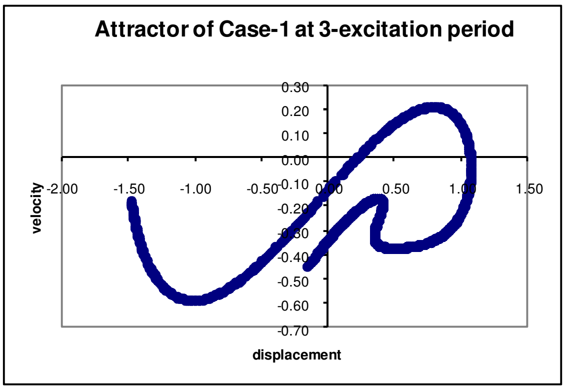


Fig. 2 (b)

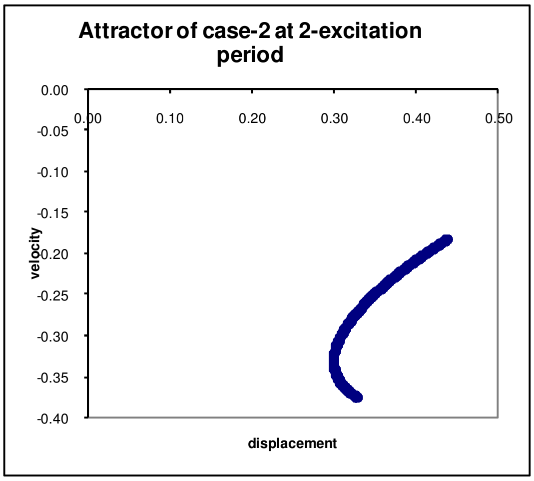


Fig. 2 (c)

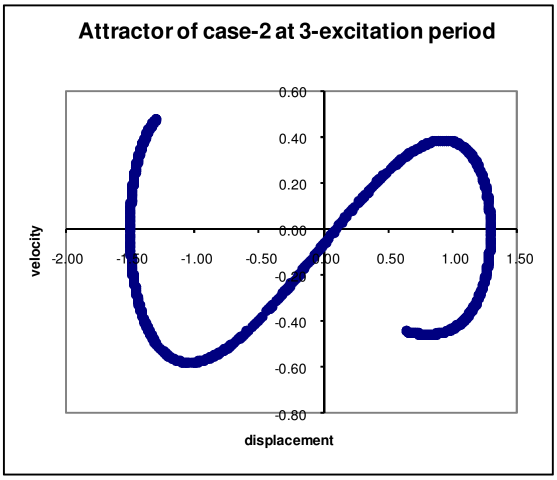


Fig. 2 (d)

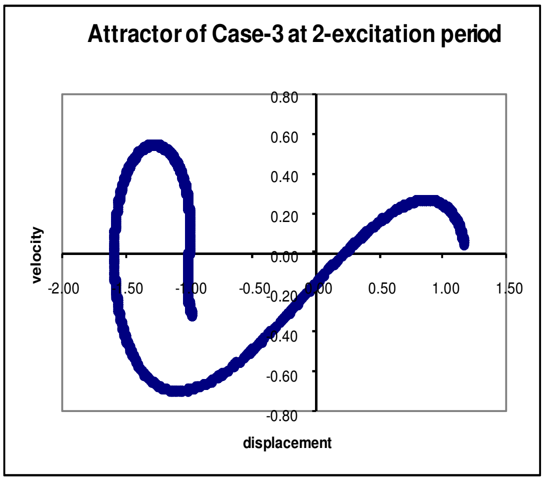


Fig. 2 (e)

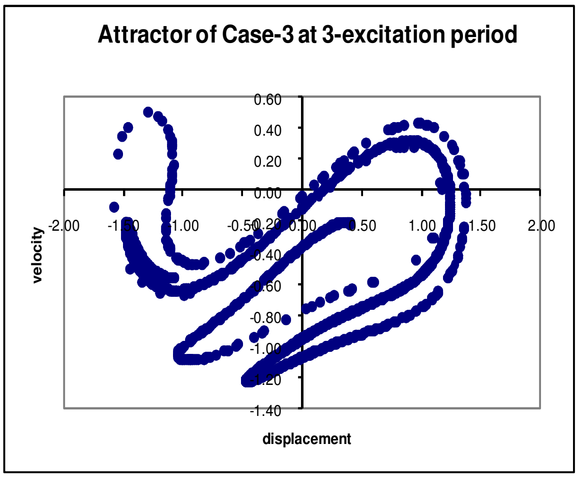


Fig. 2 (f)

Figure 2: Comparison of attractors at 2 and 3 excitation periods.

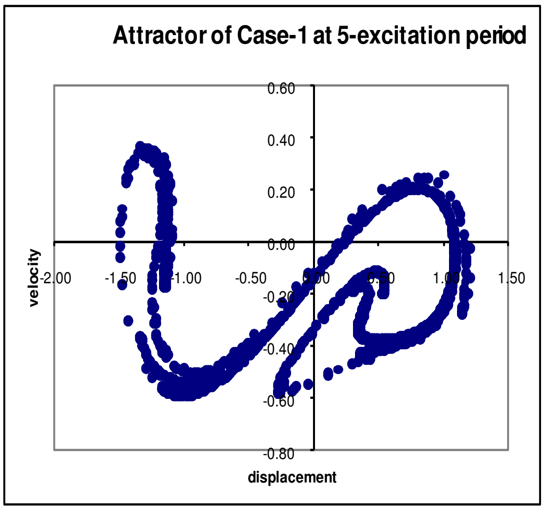


Fig. 3 (a)

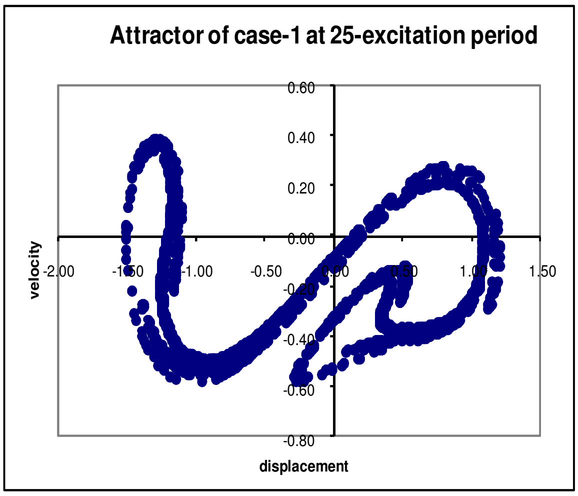


Fig. 3 (b)

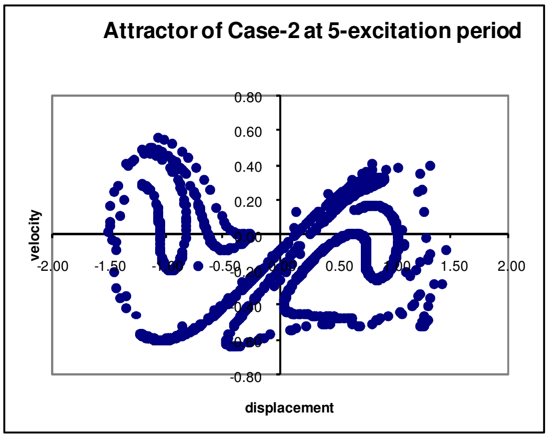


Fig. 3 (c)

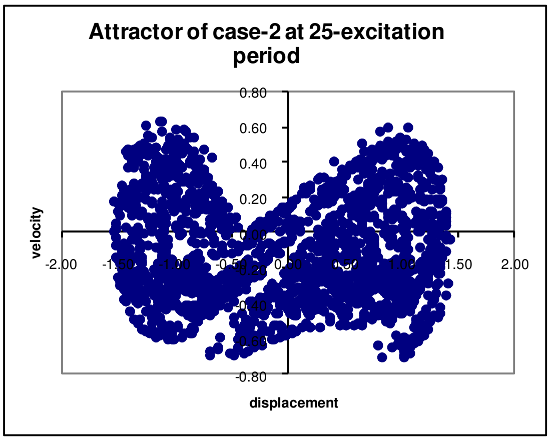
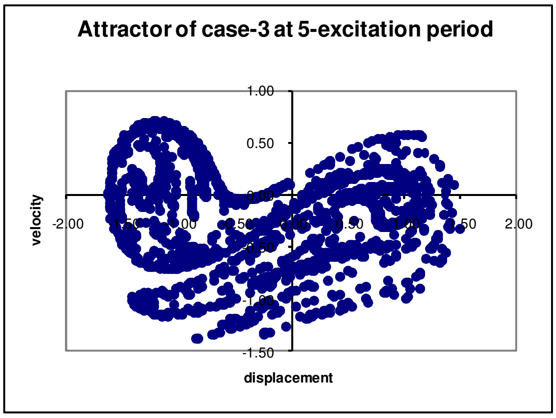


Fig. 3 (d)



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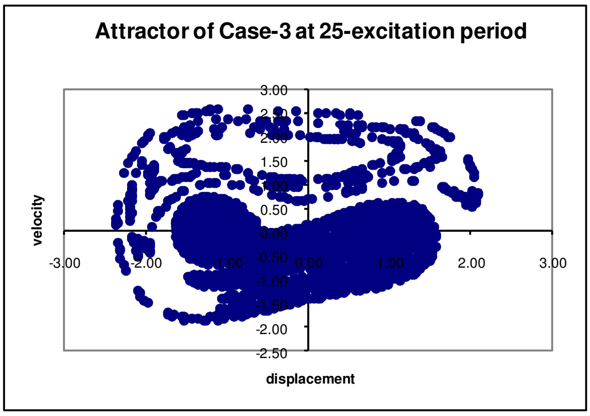

Fig. 3 (f)

Figure 3: Comparison of attractors at 5 and 25 excitation periods.

Referring to figures 1, 2 and 3 the geometrical complexity of the attractors varied widely with cases and number of excitation periods. This is an affirmation of high sensitivity to initial conditions of Duffing oscillator behaviour if excited harmonically by some parameters combinations. The attractors of Case-1 and Case-2 approach qualitatively their respective stroboscopic-ally obtained Poincare section with increasing excitation period.

The varied geometrical complexity of the attractors presented in figures 1, 2, and 3 can be characterized using fractal disk dimension measure. The algorithms for estimating the fractal disk dimension is demonstrated through presentation in table 2 and figure 4.

Table 2: Disk required for complete cover of Case-1 attractor (Poincare section) at the end of 25 excitation periods.

Disk scale	Optimum Disk counted	Disk counted in five (5) trials				
		1	2	3	4	5
1	2	3	**2**	2	2	2
2	4	5	**4**	4	5	4
3	6	8	**6**	8	8	8
4	11	14	12	12	12	**11**
5	17	19	18	18	**17**	17
6	21	**21**	21	22	21	21
7	25	**25**	28	27	26	28
8	28	31	31	**28**	30	31
9	34	38	37	**34**	39	37
10	40	**40**	42	45	41	43

11	45	47	47	49	46	**45**
12	52	54	53	55	**52**	54
13	60	**60**	62	61	60	62
14	61	65	65	67	64	**61**
15	68	72	69	69	72	**68**

Table 2 refers physical disk size for disk scale number one (1) is the largest while disk scale number fifteen (15) is the smallest. The first appearances of the optimum disk counted in five independent trials are shown in bold face through the fifteen scales of examination. Thus the optimum disk counted increases with decreasing physical disk size. The slope of line of best fit to logarithm plots of corresponding disk scale number and optimum disk counted gives the estimated fractal disk dimension of the attractor. Referring to figure 4 the estimated fractal dimension of the attractor of Case-1 at the end of 25-excitation periods is 1.3657 with an R^2 value of 0.9928.

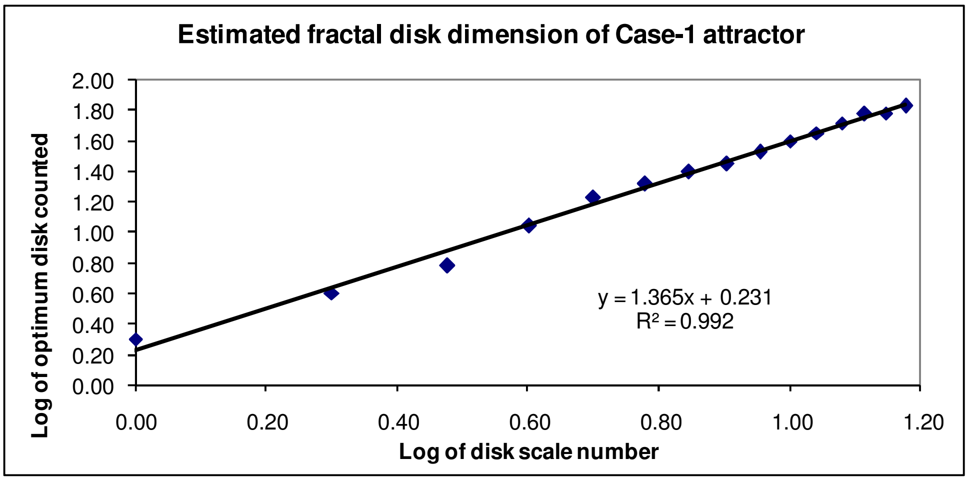

Figure 4: Fractal disk dimension of case-1 attractor at the end of 25 excitation periods.

The variation of estimated fractal disk dimension of attractors for studied cases with increasing excitation period is given in figure 5.

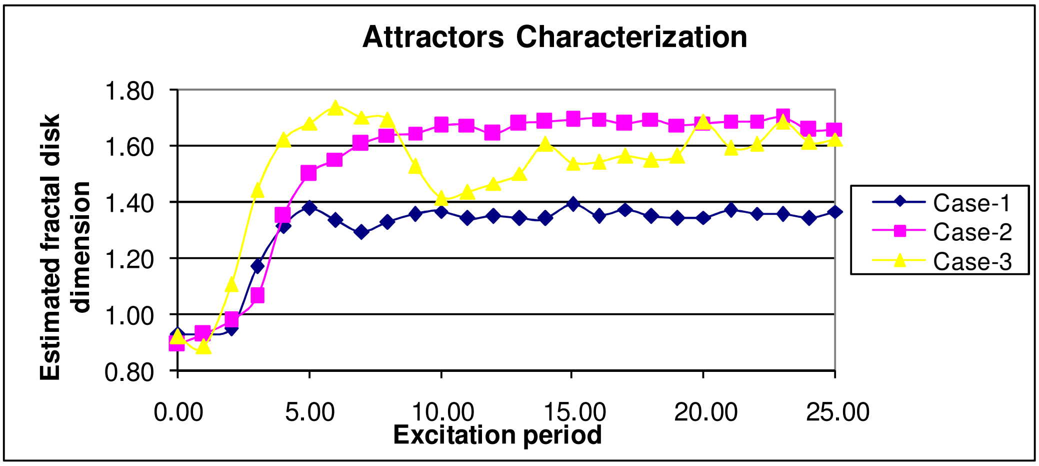

Figure 5: Variation of estimated disk dimension of attractors with excitation period.

Figure 5 refers a rise to average steady value of estimated fractal disk dimension was observed for all studied cases except Case-3. This observation with Case-3 may be due to its low damping value ($\gamma = 0.0168$) and relative very high excitation amplitude ($P_o = 0.21$). The attractor highest estimated fractal disk dimension of 1.393, 1.701 and 1.737 was recorded for the first time at corresponding excitation periods of 15, 23 and 5 for Case-1, Case-2 and Case-3 respectively.

Table 3: Estimated fractal disk dimension of Case-1 attractors at the end of 26-different excitation periods.

Standard deviation	Excitation period	Case-1 attractor different estimated fractal disk dimensions						
		Optimum	Average	Five different trials				
				1	2	3	4	5
0.02	0	0.928	0.903	0.898	0.896	0.878	0.924	0.919
0.02	1	0.927	0.918	0.917	0.908	0.920	0.888	0.956
0.01	2	0.948	0.938	0.929	0.949	0.956	0.933	0.923
0.03	3	1.170	1.146	1.128	1.161	1.137	1.121	1.182
0.03	4	1.314	1.259	1.262	1.285	1.205	1.261	1.284
0.03	5	1.376	1.340	1.351	1.348	1.308	1.315	1.380
0.02	6	1.333	1.305	1.275	1.315	1.317	1.293	1.327
0.01	7	1.292	1.297	1.307	1.304	1.292	1.293	1.290
0.01	8	1.325	1.327	1.331	1.344	1.312	1.323	1.328
0.02	9	1.355	1.341	1.358	1.309	1.351	1.332	1.357
0.02	10	1.368	1.333	1.319	1.377	1.331	1.323	1.317
0.02	11	1.341	1.324	1.323	1.350	1.348	1.295	1.306
0.02	12	1.350	1.335	1.309	1.326	1.369	1.349	1.320
0.02	13	1.344	1.341	1.330	1.357	1.361	1.348	1.310
0.02	14	1.339	1.314	1.330	1.296	1.282	1.333	1.328
0.03	15	1.394	1.345	1.324	1.324	1.325	1.400	1.351
0.02	16	1.350	1.332	1.309	1.324	1.348	1.361	1.320
0.02	17	1.374	1.345	1.361	1.356	1.362	1.327	1.320
0.03	18	1.349	1.332	1.313	1.356	1.371	1.332	1.290
0.01	19	1.343	1.341	1.325	1.357	1.341	1.333	1.352
0.06	20	1.346	1.319	1.335	1.357	1.356	1.216	1.331
0.04	21	1.368	1.340	1.344	1.355	1.270	1.341	1.390
0.02	22	1.359	1.342	1.355	1.318	1.319	1.344	1.375
0.05	23	1.356	1.323	1.342	1.331	1.335	1.362	1.242
0.02	24	1.342	1.331	1.329	1.358	1.305	1.315	1.345
0.06	25	1.366	1.331	1.229	1.383	1.361	1.325	1.356

Table 4: Estimated fractal disk dimension of Case-2 attractors at the end of 26-different excitation periods.

Standard deviation	Excitation period	Case-2 attractor different estimated fractal disk dimensions						
		Optimum	Average	Five different trials				
				1	2	3	4	5
0.01	0	0.889	0.910	0.924	0.909	0.898	0.896	0.923
0.03	1	0.926	0.906	0.920	0.890	0.894	0.883	0.944
0.06	2	0.975	0.948	0.955	0.984	0.845	0.976	0.979
0.01	3	1.063	1.058	1.063	1.059	1.057	1.041	1.071
0.02	4	1.347	1.326	1.330	1.334	1.308	1.353	1.308
0.02	5	1.499	1.463	1.481	1.495	1.452	1.449	1.437
0.02	6	1.552	1.528	1.513	1.549	1.515	1.540	1.520
0.05	7	1.605	1.558	1.567	1.621	1.554	1.480	1.571
0.01	8	1.638	1.609	1.596	1.605	1.598	1.617	1.626
0.02	9	1.646	1.630	1.626	1.653	1.601	1.643	1.629
0.02	10	1.669	1.636	1.616	1.666	1.622	1.627	1.647
0.02	11	1.674	1.648	1.667	1.650	1.621	1.651	1.650
0.01	12	1.644	1.646	1.630	1.657	1.642	1.642	1.656
0.03	13	1.678	1.653	1.669	1.690	1.631	1.637	1.639
0.02	14	1.683	1.658	1.671	1.658	1.658	1.626	1.676
0.02	15	1.691	1.664	1.669	1.650	1.702	1.661	1.639
0.01	16	1.697	1.671	1.665	1.685	1.659	1.670	1.673
0.01	17	1.679	1.664	1.675	1.653	1.652	1.653	1.683
0.05	18	1.696	1.657	1.695	1.682	1.577	1.680	1.654
0.01	19	1.675	1.655	1.653	1.657	1.655	1.641	1.667
0.02	20	1.682	1.669	1.676	1.659	1.635	1.683	1.694
0.02	21	1.688	1.675	1.681	1.707	1.674	1.667	1.648

0.02	22	1.688	1.664	1.637	1.664	1.664	1.661	1.695
0.05	23	1.701	1.656	1.712	1.583	1.643	1.681	1.661
0.01	24	1.656	1.656	1.665	1.660	1.664	1.630	1.661
0.01	25	1.660	1.652	1.665	1.660	1.647	1.650	1.636

Table 5: Estimated fractal disk dimension of Case-3 attractors at the end of 26-different excitation periods.

Standard Deviation	Excitation period	Case-3 attractor different estimated fractal disk dimensions						
		Optimum	Average	Five different trials				
				1	2	3	4	5
0.03	0	0.917	0.915	0.895	0.905	0.889	0.945	0.943
0.01	1	0.881	0.892	0.895	0.898	0.910	0.883	0.875
0.02	2	1.107	1.094	1.122	1.079	1.108	1.090	1.073
0.02	3	1.441	1.434	1.457	1.430	1.436	1.415	1.434
0.02	4	1.619	1.593	1.597	1.620	1.584	1.583	1.584
0.05	5	1.682	1.615	1.592	1.654	1.683	1.589	1.558
0.04	6	1.737	1.648	1.623	1.635	1.628	1.724	1.631
0.05	7	1.704	1.636	1.564	1.671	1.651	1.619	1.675
0.04	8	1.695	1.598	1.536	1.610	1.601	1.632	1.609
0.04	9	1.527	1.453	1.434	1.433	1.429	1.530	1.440
0.02	10	1.415	1.408	1.380	1.411	1.423	1.419	1.409
0.03	11	1.432	1.410	1.361	1.421	1.412	1.422	1.434
0.04	12	1.467	1.432	1.470	1.409	1.385	1.440	1.458
0.02	13	1.504	1.495	1.506	1.508	1.497	1.494	1.469
0.04	14	1.605	1.514	1.510	1.501	1.505	1.576	1.478
0.04	15	1.540	1.486	1.495	1.457	1.437	1.543	1.496
0.05	16	1.541	1.490	1.465	1.445	1.461	1.541	1.536
0.02	17	1.562	1.545	1.552	1.543	1.508	1.554	1.566
0.01	18	1.551	1.538	1.548	1.556	1.528	1.530	1.529
0.03	19	1.565	1.536	1.489	1.536	1.543	1.548	1.566
0.04	20	1.683	1.571	1.634	1.545	1.565	1.565	1.545
0.02	21	1.592	1.561	1.574	1.564	1.564	1.528	1.575
0.02	22	1.606	1.590	1.617	1.577	1.606	1.569	1.581
0.06	23	1.687	1.599	1.586	1.603	1.576	1.695	1.534
0.01	24	1.614	1.603	1.584	1.618	1.610	1.599	1.603
0.03	25	1.623	1.576	1.607	1.556	1.606	1.525	1.584

Tables 3, 4 and 5 refers the variation of optimum estimated fractal disk dimension with increasing excitation period is shown in figure 5.

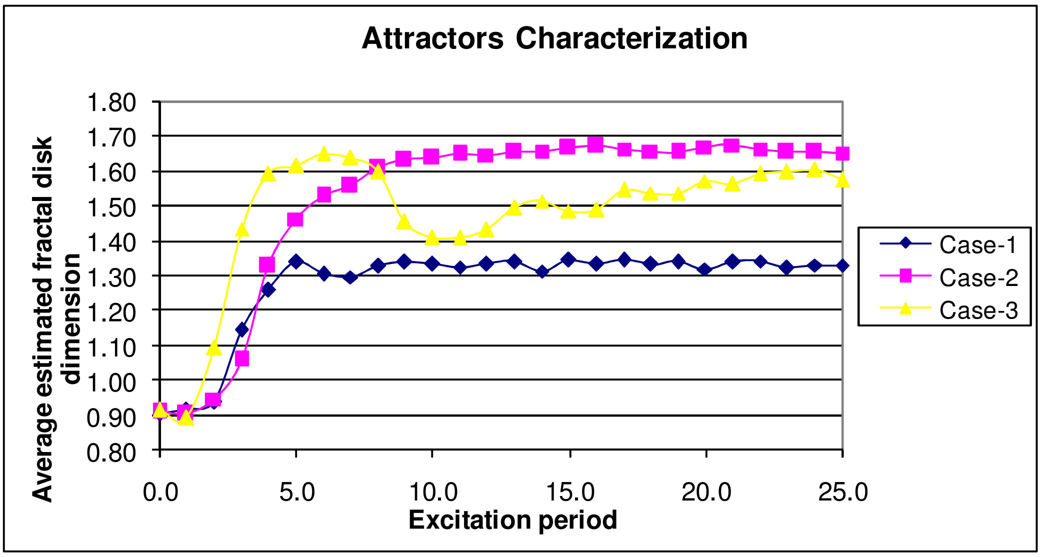

Figure 6: Variation of average estimated fractal disk dimension of attractors with excitation period.

In addition the variation of average estimated fractal disk dimension based on five independent trials with increasing excitation period is shown in figure 6. Figures 5 and 6 are same qualitatively. However the average estimated fractal disk dimensions are consistently lower than the corresponding optimum estimated fractal disk dimension for all attractors characterized. Standard deviation estimated for five trial results lies between minimum of 0.01 and maximum of 0.06 for all the cases and the attractors.

Figures 5 and 6 indicated that the attractors for different cases ultimately evolve gradually to steady geometric structure.

IV. CONCLUSIONS

The study has demonstrated the Duffing oscillator high sensitivity behaviour to set of very close initial conditions under the combination of some harmonic excitation parameters. Cases 1 and 2 evolve gradually to unique attractors which are comparable to corresponding Poincare sections obtained in the literature. On the final note, this study establishes the utility of fractal dimension as effective characterization tool and a novel alternative computational method that is faster, accurate and reliable for generating Duffing attractors or Poincare sections.

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AUTHORS BIOGRAPHY

SALAU Tajudeen Abiola Ogunniyi is a senior Lecturer in the department of Mechanical of Engineering, University of Ibadan, Nigeria. He joined the services of the University of Ibadan in February 1993 as Lecturer II in the department of Mechanical Engineering. By virtue of hard work, he was promoted to Lecturer 1 in 2002 and senior Lecturer in 2008.He had served the department in various capacities. He was the coordinator of the department for 2004/2005 and 2005/2006 Academic sessions. He was the recipient of M.K.O Abiola postgraduate scholarship in 1993/1994 academic session while on his Ph.D research programme in the University of Ibadan. Salau has many publications in learned journals and international conference proceedings especially in the area of nonlinear dynamics. He had served as external examiner in departments of Mechanical Engineering of some institutions of higher learning in the country and a reviewer/rapporteur in some reputable international conference proceedings. His area of specialization is solid mechanics with bias in nonlinear dynamics and chaos. Salau is a corporate member, Nigerian Society of Engineers (NSE). He is a registered Engineer by the council for Regulations of engineering in Nigeria.(COREN). He is happily married and blessed with children.

AJIDE Olusegun Olufemi is currently a Lecturer II in the department of Mechanical Engineering, University of Ibadan, Nigeria. He joined the services of the University of Ibadan on 1st December 2010 as Lecturer II. He had worked as the Project Site Engineer/Manager of PRETOX Engineering Nigeria Ltd, Nigeria. Ajide obtained B.Sc (Hons.) in 2003 from the Obafemi Awolowo University, Nigeria and M.Sc in 2008 from the University of Ibadan, Nigeria. He received the prestigious Professor Bamiro Prize (Vice Chancellor Award) in 2008 for the overall best M.Sc student in Mechanical Engineering (Solid Mechanics), University of Ibadan, Nigeria. He has some publications in learned journals and conference proceedings. His research interests are in area of Solid Mechanics, Applied Mechanics and Materials Engineering. Ajide is a COREN registered Engineer. He is a corporate member of the Nigerian Society of Engineers (NSE) as well as corporate member of the Nigerian Institution of Mechanical Engineers (NIMechE).